ECE 4213/5213
Test 2

Thursday, December 4, 2008
12:00 PM - 1:15 PM

Fall 2008 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ______
2. (25/20) ______
3. (25/20) ______
4. (25/20) ______
5. (25/20) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ______________ Date: ______________

1. **25/20 pts.** A causal discrete-time LTI system G has transfer function

$$G(z) = \frac{(1 - \frac{3}{2}z^{-1})(1 + \frac{1}{3}z^{-1})(1 + \frac{5}{3}z^{-1})}{(1 - z^{-1})^2(1 - \frac{1}{4}z^{-1})}.$$

(a) **5/4 pts.** Is the system G BIBO stable? (Justify your answer)

For a causal LTI discrete-time system to be BIBO stable, all the poles must be inside the unit circle of the z-plane.

Here, we have a pole at $z=1$ on the unit circle. So this system is not BIBO stable.

(b) **5/4 pts.** Does the system G have minimum phase? (Justify your answer)

This system has a zero at $z=\frac{3}{2}$, which is outside the unit circle of the z-plane.

Since a minimum phase system must have all zeros inside the unit circle, this system is not minimum phase.

Since there is also a zero at $z=-\frac{1}{3}$ inside the unit circle, this system in fact has mixed phase.

(c) **15/12 pts.** Now consider a second causal discrete-time LTI system H with impulse response

$$h[n] = a^n g[n],$$

where $a \in \mathbb{R}$ is a real constant and where $g[n]$ is the impulse response of the system G from part (a). For what values of the real constant a is the system causal, BIBO stable, and minimum phase?

System G is given to be causal, so $g[n]=0$ $\forall n<0$. This implies that $h[n]=a^n g[n]=0$ $\forall n<0$, so H is also causal. Therefore H will be causal, BIBO stable, and minimum phase if all the poles and zeros of $H(z)$ are inside the unit circle of the z-plane.

Table: $a^n g[n] \overset{Z}{\longleftrightarrow} G\left(\frac{z}{a}\right)$. So,

$$H(z) = G\left(\frac{z}{a}\right) = \frac{\left(1-\frac{3a}{2}z^{-1}\right)\left(1+\frac{a}{3}z^{-1}\right)\left(1+\frac{5a}{3}z^{-1}\right)}{\left(1-az^{-1}\right)^2\left(1-\frac{a}{4}z^{-1}\right)}$$

The poles are at $z=a$ and $z=\frac{a}{4}$. The zeros are at $z=\frac{3a}{2}$, $z=-\frac{a}{3}$, and $z=-\frac{5a}{3}$. These will all be inside the unit circle if

$$\left|-\frac{5a}{3}\right| < 1$$

$$5|a| < 3$$

$$\boxed{|a| < \frac{3}{5}}$$

2. **25/20 pts**. Consider the impulse response $h_{HP}[n]$ of the ideal highpass filter H_{HP}, as shown in Eq. (10.16) of the text.

   (a) **10/8 pts**. Truncate the ideal impulse response (10.16) to a length of $2M+1$ samples, $M \in \mathbb{Z}$, and shift to the right to obtain a causal FIR highpass impulse response $\widehat{h}_{HP}[n]$. Give an explicit expression for $\widehat{h}_{HP}[n]$.

The required impulse response $\hat{h}_{HP}[n]$ is obtained by first truncating the ideal impulse response so that it is zero outside the interval $-M \le n \le M$, and then replacing "n" with "$n-M$" to shift right by M samples to make the filter causal. This gives us

$$\hat{h}_{HP}[n] = \begin{cases} 1 - \frac{\omega_c}{\pi}, & n = M, \\ -\frac{\sin[\omega_c(n-M)]}{\pi(n-M)}, & n \neq M \text{ and } 0 \le n \le 2M, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 2 cont....

(b) **15/12 pts**. Show that the causal FIR filter $\widehat{h}_{HP}[n]$ from part (a) and the causal lowpass FIR filter $\widehat{h}_{LP}[n]$ given in Eq. (10.15) of the text are a delay complementary pair.

According to (7.90) on p. 391 of the text, $\hat{H}_{HP}$ and $\hat{H}_{LP}$ are a delay complementary pair if $\exists$ constants $n_0 \in \mathbb{Z}$ and $\gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and $\hat{H}_{HP}(z) + \hat{H}_{LP}(z) = \gamma z^{-n_0}$. (*)

Now, $\hat{H}_{HP}(z) + \hat{H}_{LP}(z) = \sum_{n=-\infty}^{\infty} \hat{h}_{HP}[n] z^{-n} + \sum_{n=-\infty}^{\infty} \hat{h}_{LP}[n] z^{-n}$

$= \sum_{n=-\infty}^{\infty} \left(\hat{h}_{HP}[n] + \hat{h}_{LP}[n] \right) z^{-n}$. (**)

Eq's. (*) and (**) establish that $\hat{H}_{HP}$ and $\hat{H}_{LP}$ are a delay complementary pair if $\hat{h}_{HP}[n] + \hat{h}_{LP}[n] = \gamma \delta[n-n_0]$ for some integer n_0 and some nonzero constant $\gamma \in \mathbb{C}$.

Now,

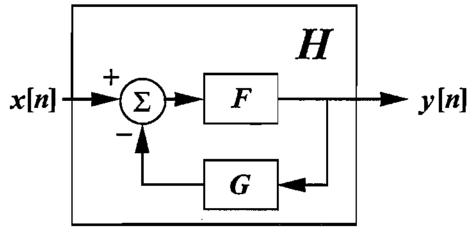
(i) if $n < 0$ or $n \geq 2M+1$, then
$\hat{h}_{HP}[n] + \hat{h}_{LP}[n] = 0 + 0 = 0$.

(ii) if $0 \leq n < M$ or $M < n \leq 2M$, then
$$\hat{h}_{HP}[n] + \hat{h}_{LP}[n] = -\frac{\sin[\omega_c(n-M)]}{\pi(n-M)} + \frac{\sin[\omega_c(n-M)]}{\pi(n-M)} = 0.$$

(iii) if $n=M$, we have by l'Hôpital's rule
$$\hat{h}_{LP}[M] = \lim_{n \to M} \frac{\frac{\partial}{\partial n} \sin[\omega_c(n-M)]}{\frac{\partial}{\partial n} \pi(n-M)} = \lim_{n \to M} \frac{\omega_c \cos[\omega_c(n-M)]}{\pi} = \frac{\omega_c}{\pi}.$$
So $\hat{h}_{HP}[M] + \hat{h}_{LP}[M] = 1 - \frac{\omega_c}{\pi} + \frac{\omega_c}{\pi} = 1$.

With (i), (ii), and (iii), we have all together that
$$\hat{h}_{HP}[n] + \hat{h}_{LP}[n] = \begin{cases} 1, & n = M \\ 0, & \text{otherwise} \end{cases} = \delta[n-M].$$
Therefore, $\hat{H}_{LP}$ and $\hat{H}_{HP}$ are a delay complementary

3. **25/20 pts.** Consider the causal LTI filter H shown in the figure below.



The impulse response of the LTI system F is given by

$$f[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1].$$

The causal LTI system G has input $x_g[n]$ and output $y_g[n]$ that are related by the difference equation

$$y_g[n] + \frac{1}{12}y_g[n-1] - \frac{1}{12}y_g[n-2] = x_g[n] - \frac{1}{4}x_g[n-1].$$

Give a Direct Form II$_t$ (transpose) realization of the system H.

Table: $F(z) = \frac{1}{2} + \frac{1}{2}z^{-1}, \quad |z| > 0.$

For G: $Y_g(z)\left[1 + \frac{1}{12}z^{-1} - \frac{1}{12}z^{-2}\right] = X_g(z)\left[1 - \frac{1}{4}z^{-1}\right]$

$$G(z) = \frac{Y_g(z)}{X_g(z)} = \frac{1 - \frac{1}{4}z^{-1}}{1 + \frac{1}{12}z^{-1} - \frac{1}{12}z^{-2}} = \frac{1 - \frac{1}{4}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 + \frac{1}{3}z^{-1})}$$

$$= \frac{1}{1 + \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$H(z) = \frac{F(z)}{1 + F(z)G(z)} = \frac{\frac{1}{2} + \frac{1}{2}z^{-1}}{1 + (\frac{1}{2} + \frac{1}{2}z^{-1})(\frac{1}{1 + \frac{1}{3}z^{-1}})} \cdot \frac{1 + \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}$$

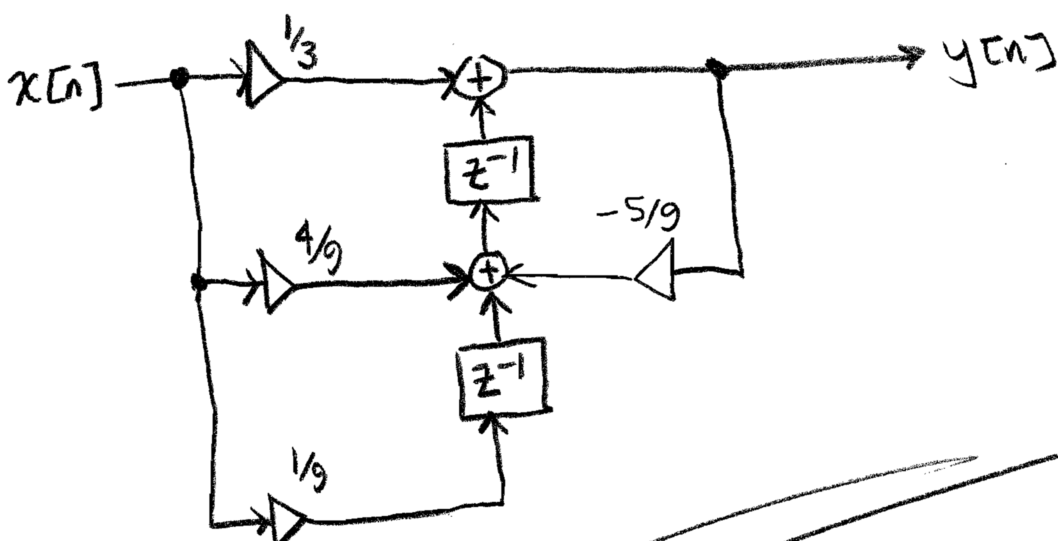
$$= \frac{(\frac{1}{2} + \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}{1 + \frac{1}{3}z^{-1} + \frac{1}{2} + \frac{1}{2}z^{-1}} = \frac{\frac{1}{2} + \frac{1}{6}z^{-1} + \frac{1}{2}z^{-1} + \frac{1}{6}z^{-2}}{\frac{3}{2} + \frac{5}{6}z^{-1}}$$

More Workspace for Problem 3...

$$\ldots\ H(z) = \frac{\frac{1}{2} + \frac{2}{3}z^{-1} + \frac{1}{6}z^{-2}}{\frac{3}{2} + \frac{5}{6}z^{-1}} \cdot \frac{\frac{2}{3}}{\frac{2}{3}} = \frac{\frac{1}{3} + \frac{4}{9}z^{-1} + \frac{1}{9}z^{-2}}{1 + \frac{5}{9}z^{-1}}, \quad |z| > \frac{5}{9}$$

The Direct Form II_t realization can be written down directly from Fig. 8.14(b) on p. 439 of the text:

$$p_0 = \frac{1}{3} \qquad p_1 = \frac{4}{9} \qquad p_2 = \frac{1}{9} \qquad d_1 = 5/9$$



4. **25/20 pts.** Use the bilinear transform to design a first-order lowpass Butterworth filter with a 3 dB cutoff frequency $\omega_c = 0.2\pi$.

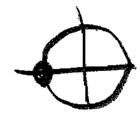
– Prewarp ω_c using (9.18) on p. 494 with $T=2$ as on p. 499 (top):

$$\Omega_c = \tan\left(\frac{\omega_c}{2}\right) = \tan(0.1\pi) = 0.324920$$

– The order is specified: $N=1$.

– Use (4.37) on p. 190:

$$p_1 = \Omega_c e^{j[\pi(1+2-1)/2]}$$



$$= \Omega_c e^{j2\pi/2} = \Omega_c e^{j\pi} = \Omega_c(-1) = -\Omega_c.$$

– Plug into (4.36) on p. 190:

$$H_a(s) = \frac{\Omega_c^1}{(s+\Omega_c)} = \frac{\Omega_c}{s+\Omega_c}$$

– Map back to z using (9.15) on p. 494 with $T=2$:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{\Omega_c}{\frac{1-z^{-1}}{1+z^{-1}} + \Omega_c} \cdot \frac{1+z^{-1}}{1+z^{-1}}$$

$$= \frac{\Omega_c(1+z^{-1})}{1-z^{-1}+(1+z^{-1})\Omega_c} = \frac{\Omega_c + \Omega_c z^{-1}}{1-z^{-1}+\Omega_c+\Omega_c z^{-1}}$$

$$= \frac{\Omega_c + \Omega_c z^{-1}}{(1+\Omega_c) - (1-\Omega_c)z^{-1}} = \frac{0.32490 + 0.32490 z^{-1}}{1.32490 - 0.67508 z^{-1}}$$

$$\cdot \frac{\frac{1}{1.32490}}{\frac{1}{1.32490}} = \boxed{\frac{0.24524 + 0.24524 z^{-1}}{1 - 0.50953 z^{-1}}}$$

More Workspace for Problem 4...

5. **25/20 pts.** Use the window design method with an appropriate **fixed** window from Table 10.2 (page 535 of the text) to design a lowpass FIR digital filter with the following specifications:

passband edge freq.	$\omega_p = 0.3\pi$ rad/sample
stopband edge freq.	$\omega_s = 0.35\pi$ rad/sample
max. passband ripple	$\delta_p = 0.01$
max. stopband ripple.	$\delta_s = 0.01$

Give the filter impulse response $h[n]$.

The minimum stopband attenuation follows from (4.30) on p. 187:

$$\alpha_s = -20\log_{10}\delta_s = 40 \text{ dB}.$$

Transition bandwidth: $\Delta\omega = \omega_s - \omega_p = 0.35\pi - 0.30\pi = 0.05\pi$.

- From table 10.2, the stopband attenuation spec can be met by a Hann, Hamming, or Blackman window. The order required for each window type can be determined from the formulas in the last column of Table 10.2:

Hann: $\Delta\omega = 0.05\pi = \frac{3.11\pi}{M} \Rightarrow M = \frac{3.11}{0.05} = 62.2$

$$\text{order} = \lceil 2M \rceil = \lceil 124.4 \rceil = 125.$$

$$\text{length} = \lceil 2M+1 \rceil = \lceil 125.4 \rceil = 126.$$

Hamming: $\Delta\omega = 0.05\pi = \frac{3.32\pi}{M} \Rightarrow M = \frac{3.32}{0.05} = 66.4$

$$\text{order} = \lceil 2M \rceil = \lceil 132.8 \rceil = 133$$

$$\text{length} = \lceil 2M+1 \rceil = \lceil 133.8 \rceil = 134$$

Blackman: $\Delta\omega = 0.05\pi = \frac{5.56\pi}{M} \Rightarrow M = \frac{5.56}{0.05} = 111.2$

$$\text{order} = \lceil 2M \rceil = \lceil 222.4 \rceil = 223$$

$$\text{length} = \lceil 2M+1 \rceil = \lceil 223.4 \rceil = 224$$

★Use Hann because it meets the spec with the lowest order.

More Workspace for Problem 5...

→ since the formulas given on p. 533 of the text require the length to be odd, we have to round up M to 63. This gives us an order of 126 with a length of $2M+1 = 127$.

From (10.30) on p. 533, the zero-phase window is

$$w[n] = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi n}{127}\right)\right], \quad -63 \le n \le 63.$$

From the text on p. 536 (between the two examples), we set the cutoff frequency to $\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.65\pi}{2} = 0.325\pi$.

From (10.14) on p. 528, the zero phase ideal impulse response is

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\sin(0.325\pi n)}{\pi n}.$$

So the zero phase windowed impulse response is

$$h_{LP}[n]\,w[n] = \frac{\sin(0.325\pi n)}{2\pi n}\left[1 + \cos\left(\frac{2\pi n}{127}\right)\right], \quad -63 \le n \le 63.$$

Finally, the causal impulse response is obtained by shifting right by $M=63$:

$$\boxed{h[n] = \frac{\sin[0.325\pi(n-63)]}{2\pi(n-63)}\left\{1 + \cos\left[\frac{2\pi(n-63)}{127}\right]\right\}, \quad 0 \le n \le 127}$$