ECE 4213/5213
Test 2

Wednesday, December 1, 2010
4:30 PM - 5:45 PM

Fall 2010 — Name: SOLUTION

Dr. Havlicek — Student Num:__________

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

1. **25/20 pts**. A causal FIR filter H has a four-point (finite length) impulse response given by

$$\begin{aligned} h[n] &= [\tfrac{1}{4}\ \tfrac{1}{4}\ \tfrac{1}{4}\ \tfrac{1}{4}] \\ &= \frac{1}{4}\delta[n] + \frac{1}{4}\delta[n-1] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n-3], \quad 0 \le n \le 3. \end{aligned}$$

The system input is a four-point (finite length) signal given by

$$\begin{aligned} x[n] &= [1\ 1\ {-1}\ 0] \\ &= \delta[n] + \delta[n-1] - \delta[n-2], \quad 0 \le n \le 3. \end{aligned}$$

Use the DFT to find the finite length system output $y[n]$.

NOTE: in this problem, you are being asked to use the DFT to implement *linear convolution*, **not** *circular convolution*.

To get linear convolution, we need to zero pad both signals to length $4+4-1 = 7$. $\underline{N=7}$.

$h_7[n] = [\tfrac{1}{4}\ \tfrac{1}{4}\ \tfrac{1}{4}\ \tfrac{1}{4}\ 0\ 0\ 0]$ $\qquad W_7 = e^{-j2\pi/7}$

$$H_7[k] = \sum_{n=0}^{6} h_7[n] W_7^{nk}$$

$$= \tfrac{1}{4} + \tfrac{1}{4}W_7^{k} + \tfrac{1}{4}W_7^{2k} + \tfrac{1}{4}W_7^{3k}$$

$x_7[n] = [1\ 1\ {-1}\ 0\ 0\ 0\ 0]$

$$X_7[k] = \sum_{n=0}^{6} x_7[n] W_7^{nk} = 1 + W_7^{k} - W_7^{2k}$$

$$Y[k] = X_7[k]\, H_7[k]$$

$$\begin{aligned} &= \tfrac{1}{4} + \tfrac{1}{4}W_7^{k} + \tfrac{1}{4}W_7^{2k} + \tfrac{1}{4}W_7^{3k} \\ &\quad + \tfrac{1}{4}W_7^{k} + \tfrac{1}{4}W_7^{2k} + \tfrac{1}{4}W_7^{3k} + \tfrac{1}{4}W_7^{4k} \\ &\quad - \tfrac{1}{4}W_7^{2k} - \tfrac{1}{4}W_7^{3k} - \tfrac{1}{4}W_4^{4k} - \tfrac{1}{4}W_7^{5k} \end{aligned}$$

2

$$= \tfrac{1}{4} + \tfrac{1}{2}W_7^{k} + \tfrac{1}{4}W_7^{2k} + \tfrac{1}{4}W_7^{3k} - \tfrac{1}{4}W_7^{5k} \longrightarrow$$

More Workspace for Problem 1...

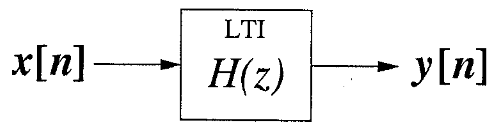
$$Y[k] = \frac{1}{4}W_7^{0k} + \frac{1}{2}W_7^{k} + \frac{1}{4}W_7^{2k} + \frac{1}{4}W_7^{3k} + 0W_7^{4k} - \frac{1}{4}W_7^{5k} + 0W_7^{6k}$$

$$= \sum_{n=0}^{6} y[n] W_7^{nk}$$

$$\Rightarrow y[n] = \left[\frac{1}{4} \;\; \frac{1}{2} \;\; \frac{1}{4} \;\; \frac{1}{4} \;\; 0 \;\; -\frac{1}{4} \;\; 0 \right]$$

$$= \frac{1}{4}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n-3] - \frac{1}{4}\delta[n-5], \quad 0 \leq n < 7.$$

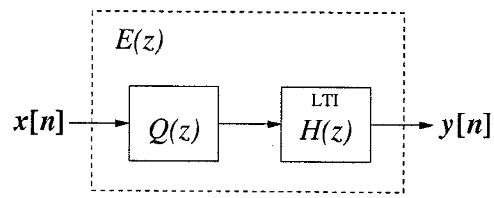
2. **25/20 pts**. Consider the non-ideal digital communications channel H shown below.



It is determined by experiment that this channel can be modeled as a **causal** IIR LTI system with transfer function

$$H(z) = \frac{\left(1 - \frac{3}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)\left(1 + \frac{5}{3}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{4}z^{-1}\right)}.$$

Design a digital pre-equalizer $Q(z)$ to go in series with the channel as shown below



so that the overall *equalized* channel $E(z)$ is **allpass**.

It is required for your pre-equalizer $Q(z)$ to be both **causal** and **stable** and to have **minimum group delay**.

We would like to take $Q(z) = 1/H(z)$. But we can't do that. The reason is that in order for $Q(z)$ to be stable, causal, and min group delay (equivalent to min phase), $Q(z)$ must have all poles and all zeros inside the unit circle. But $H(z)$ has zeros at $z = \frac{3}{2}$ and $z = -5/3$ that are outside the unit circle. So we need to find $H_{min}(z)$ such that $|H_{min}(e^{j\omega})| = |H(e^{j\omega})|$ $\forall \omega \in \mathbb{R}$ and $H_{min}(z)$ has all poles and zeros inside the unit circle.

Since the poles of $H(z)$ are already inside the unit circle, we need to factor $H(z)$ as

$$H(z) = H_{min}(z) H_{ap}(z)$$

where the bad zeros in $H(z)$ are reflected inside the unit circle in $H_{min}(z)$

More Workspace for Problem 2...

$$H(z) = \frac{(1-\frac{3}{2}z^{-1})(1+\frac{1}{3}z^{-1})(1+\frac{5}{3}z^{-1})}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})}$$

$$= \frac{(1+\frac{1}{3}z^{-1})}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})} \quad (1-\frac{3}{2}z^{-1})(1+\frac{5}{3}z^{-1})$$

$$= \frac{(1+\frac{1}{3}z^{-1})}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})} \quad (1-\frac{3}{2}z^{-1})\frac{z^{-1}-\frac{3}{2}}{z^{-1}-\frac{3}{2}} \quad (1+\frac{5}{3}z^{-1})\frac{z^{-1}+\frac{5}{3}}{z^{-1}+5/3}$$

$$= \underbrace{\frac{(1+\frac{1}{3}z^{-1})(z^{-1}-\frac{3}{2})(z^{-1}+\frac{5}{3})}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})}}_{H_{min}(z)} \cdot \underbrace{\frac{(1-\frac{3}{2}z^{-1})(1+5/3\,z^{-1})}{(z^{-1}-\frac{3}{2})(z^{-1}+5/3)}}_{\text{ALL PASS}}$$

$$H_{min}(z) = \frac{(1+\frac{1}{3}z^{-1})(-\frac{3}{2})(1-\frac{2}{3}z^{-1})(\frac{5}{3})(1+\frac{3}{5}z^{-1})}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})}$$

$$= -\frac{5}{2}\,\frac{(1+\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})(1+\frac{3}{5}z^{-1})}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})}$$

$$\boxed{Q(z) = \frac{1}{H_{min}(z)} = -\frac{2}{5}\,\frac{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})}{(1+\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})(1+\frac{3}{5}z^{-1})}, \quad |z| > \frac{2}{3}}$$

3. **25/20 pts.** An FIR linear phase filter has a **real-valued** impulse response $h[n]$ that is nonzero only for $0 \leq n \leq 6$. In addition, it is known that $h[0] = 1$.

The transfer function $H(z)$ has a complex zero at $z = 0.4e^{-j\pi/3}$ and a real zero at $z = 3$. **Note:** these are not the *only* zeros.

Find $H(z)$. We are given that $h[n]$ has length 7, so $H(z)$ is 6^{th} order in z^{-1} $\Rightarrow$ $H(z)$ has six zeros. Also, since H is FIR, the poles are all at $z=0$. So $H(z) = A\prod_{k=1}^{6}(1-z_k z^{-1})$ where $z_1 \cdots z_6$ are the six zeros and A is a constant that ensures $h[0]=1$ by solving

$$H(z) = A\prod_{k=1}^{6}(1-z_k z^{-1}) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots + h[6]z^{-6}.$$

→ Because $h[n]$ is real, complex zeros in $H(z)$ must occur in complex conjugate pairs.

→ Because H is linear phase, a zero at $z=z_0$ must be accompanied by another zero at $z = 1/z_0$.

$\Rightarrow$ $z_1 = \frac{4}{10}e^{-j\pi/3}$ must be accompanied by $z_2 = \frac{1}{z_1} = \frac{10}{4}e^{j\pi/3} = \frac{5}{2}e^{j\pi/3}$

$z_1 = \frac{4}{10}e^{-j\pi/3}$ must be accompanied by $z_3 = z_1^* = \frac{2}{5}e^{j\pi/3}$

z_3 must be accompanied by $z_4 = \frac{1}{z_3} = \frac{5}{2}e^{-j\pi/3}$

$z_5 = 3$ must be accompanied by $z_6 = \frac{1}{z_5} = \frac{1}{3}$.

$z_1 = \frac{2}{5}e^{-j\pi/3}$; $z_2 = \frac{5}{2}e^{j\pi/3}$; $z_3 = \frac{2}{5}e^{j\pi/3}$; $z_4 = \frac{5}{2}e^{-j\pi/3}$; $z_5 = 3$; $z_6 = \frac{1}{3}$

$$H(z) = A\left(1-\tfrac{2}{5}e^{-j\pi/3}z^{-1}\right)\left(1-\tfrac{5}{2}e^{j\pi/3}z^{-1}\right)\left(1-\tfrac{2}{5}e^{j\pi/3}z^{-1}\right)\left(1-\tfrac{5}{2}e^{-j\pi/3}z^{-1}\right)$$
$$\times\left(1-3z^{-1}\right)\left(1-\tfrac{1}{3}z^{-1}\right)$$

The coefficient of $z^0 = 1$ is $A \cdot 1^6 = A = h[0] = 1 \Rightarrow A = 1$.

$$H(z) = \left(1-\tfrac{2}{5}e^{-j\pi/3}\right)\left(1-\tfrac{5}{2}e^{j\pi/3}z^{-1}\right)\left(1-\tfrac{2}{5}e^{j\pi/3}z^{-1}\right)\left(1-\tfrac{5}{2}e^{-j\pi/3}z^{-1}\right)\left(1-3z^{-1}\right)\left(1-\tfrac{1}{3}z^{-1}\right)$$

4. **25/20 pts**. A causal discrete-time system G has transfer function

$$G(z) = \frac{5 + z^{-1}}{(3 - z^{-1})(4 - z^{-1})}.$$

Give a Parallel Form I realization of $G(z)$.

Perform a PFE in z^{-1}:

$$\frac{5+\theta}{(3-\theta)(4-\theta)} = \frac{A}{3-\theta} + \frac{B}{4-\theta}$$

$$A = \left.\frac{5+\theta}{4-\theta}\right|_{\theta=3} = 8$$

$$B = \left.\frac{5+\theta}{3-\theta}\right|_{\theta=4} = -9$$

$$G(z) = \frac{8}{3-z^{-1}} - \frac{9}{4-z^{-1}} = \frac{8/3}{1-\frac{1}{3}z^{-1}} - \frac{9/4}{1-\frac{1}{4}z^{-1}}$$

$$= \sum_{l=1}^{2} \frac{\rho_l}{1-\lambda_l z^{-1}} \qquad (6.39) \text{ p. 315 of text}$$

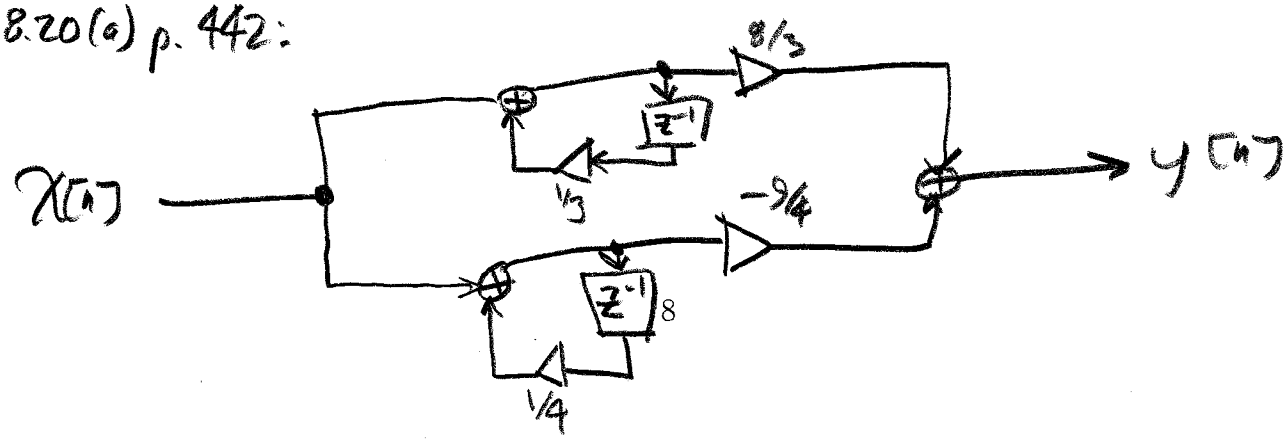
$$\rho_1 = 8/3 \quad \rho_2 = -9/4$$
$$\lambda_1 = 1/3 \quad \lambda_2 = 1/4$$

In terms of (8.30) on p. 442 of the text, we have

$$G(z) = \gamma_0 + \sum_{k=1}^{2} \frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

$$\gamma_0 = 0 \quad \gamma_{01} = 8/3 \quad \gamma_{01} = 0 \quad \gamma_{02} = -9/4 \quad \gamma_{12} = 0$$
$$\alpha_{11} = -1/3 \quad \alpha_{21} = 0 \quad \alpha_{12} = -\frac{1}{4} \quad \alpha_{22} = 0$$

Fig. 8.20(a) p. 442:



5. **25/20 pts.** Design an **analog** Type I Chebyshev low pass filter to meet the following **analog** specification:

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 9/10$$
$$\sqrt{1+\varepsilon^2} = 10/9$$
$$1+\varepsilon^2 = 100/81$$
$$\varepsilon^2 = \frac{100-81}{81} = 19/81$$
$$\varepsilon = 0.48432$$

passband edge freq.	$\Omega_p = \pi$ rad/sec
stopband edge freq.	$\Omega_s = 2\pi$ rad/sec
max. stopband ripple	$1/A = 1/\pi$
passband equiripple	$1/\sqrt{1+\varepsilon^2} = 0.9$

Give the analog filter transfer function $H_a(s)$.

Hint: The design formulas for the analog Type I Chebyshev filter are given on pages 191 and 192 of the text.

(4.43): $$N = \left\lceil \frac{\cosh^{-1}\left(\sqrt{A^2-1}/\varepsilon\right)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{2.50279}{1.31696} \right\rceil = \lceil 1.90043 \rceil = 2$$

(4.45b): $$\gamma = \left[\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right]^{1/2} = 2.08780 \quad \zeta = \frac{\gamma^2+1}{2\gamma} = 1.28339$$

$$\xi = \frac{\gamma^2-1}{2\gamma} = 0.804412$$

(4.45a): $$\sigma_1 = -\pi\xi \sin \pi/4 = -1.78695; \quad \Omega_1 = \pi\zeta \cos\frac{\pi}{4} = 2.85097$$
$$\sigma_2 = -\pi\xi \sin 3\pi/4 = -1.78695; \quad \Omega_2 = \pi\zeta\cos\frac{3\pi}{4} = -2.85097$$

(4.44): $$p_1 = \sigma_1 + j\Omega_1 = -1.78695 + j2.85097$$
$$p_2 = \sigma_2 + j\Omega_2 = -1.78695 - j2.85097$$

Notes p. 5-16B: $$H_a(s) = C_0 \prod_{\ell=1}^{N} \frac{-p_\ell}{s-p_\ell}; \quad N \text{ even: } C_0 = \frac{1}{\sqrt{1+\varepsilon^2}} = 0.9$$

$$H_a(s) = 0.9 \frac{(1.78695 - j2.85097)}{s+(1.78695-j2.85097)} \frac{(1.78695+j2.85097)}{s+(1.78695+j2.85097)}$$

$$= \frac{0.9(3.19319 + 8.12803)}{s^2 + 3.57390s + (3.19319 + 8.12803)}$$

$$\boxed{H_a(s) = \frac{10.1891}{s^2 + 3.57390s + 11.3212}}$$