ECE 4213/5213
Test 2

Monday, November 28 - Friday, December 2, 2011
This test is **DUE** at 5:00 PM on Friday, December 2, 2011

Fall 2011 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: This is an open book, open notes, **takehome** test. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 96 hours to complete the test. Do not discuss this test with other students. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: Date:

1. **25/20 pts**. H is a causal FIR digital filter with impulse response

$$\begin{aligned} h[n] &= [-1 \quad 2 \quad -1] \\ &= -\delta[n] + 2\delta[n-1] - \delta[n-2], \quad 0 \le n \le 2. \end{aligned}$$

The input is a length-4 digital signal $x[n]$ given by

$$\begin{aligned} x[n] &= [1 \quad 2 \quad 3 \quad -1] \\ &= \delta[n] + 2\delta[n-1] + 3\delta[n-2] - \delta[n-3], \quad 0 \le n \le 3. \end{aligned}$$

Use the DFT to find the finite length system output $y[n]$.

NOTE: in this problem, you are being asked to use the DFT to implement *linear convolution*, **not** *circular convolution*.

For linear convolution, zero pad to length $3+4-1=6$

$h_6[n] = [-1 \quad 2 \quad -1 \quad 0 \quad 0 \quad 0] \quad H_6[k] = -1 + 2W_6^k - W_6^{2k}$

$x_6[n] = [1 \quad 2 \quad 3 \quad -1 \quad 0 \quad 0 \quad 0] \quad X_6[k] = 1 + 2W_6^k + 3W_6^{2k} - W_6^{3k}$

$$\begin{aligned} Y_6[k] = H_6[k]X_6[k] &= -X_6[k] + 2W_6^k X_6[k] - W_6^{2k} X_6[k] \\ &= -1 - 2W_6^k - 3W_6^{2k} + W_6^{3k} \\ &\quad + 2W_6^k + 4W_6^{2k} + 6W_6^{3k} - 2W_6^{4k} \\ &\quad - W_6^{2k} - 2W_6^{3k} - 3W_6^{4k} + W_6^{5k} \end{aligned}$$

$Y_6[k] = -1 + 0W_6^k + 0W_6^{2k} + 5W_6^{3k} - 5W_6^{4k} + W_6^{5k}$

But by definition, $Y_6[k] = \sum_{n=0}^{5} y_6[n] W_6^{nk}$

$= y_6[0] + y_6[1]W_6^k + y_6[2]W_6^{2k} + \dots + y_6[5]W_6^{5k}$

$\boxed{\text{So } y_6[n] = [-1 \quad 0 \quad 0 \quad 5 \quad -5 \quad 1]}$

More Workspace for Problem 1...

2. **25/20 pts.** H is a causal, stable 4th-order linear phase FIR filter ($N = 4$). The transfer function $H(z)$ has a zero at $z = 1 + j$. The DC gain of the filter is given by

$$H(e^{j0}) = H(1) = 0.5.$$

(a) **9/7 pts.** Find the transfer function $H(z)$ and give a pole-zero plot.

We are given the zero $z_1 = 1+j$. Because H is a real-coefficient linear phase FIR filter, there are three more zeros at

$$z_2 = z_1^* = 1-j$$

$$z_3 = 1/z_1 = \frac{1}{1+j}\,\frac{1-j}{1-j} = \frac{1-j}{2} = \frac{1}{2} - j\frac{1}{2}$$

$$z_4 = 1/z_2 = \frac{1}{1-j}\,\frac{1+j}{1+j} = \frac{1+j}{2} = \frac{1}{2} + j\frac{1}{2}.$$

So $H(z) = C_0(1-z_1 z^{-1})(1-z_2 z^{-1})(1-z_3 z^{-1})(1-z_4 z^{-1})$

$$= C_0\left[1-(1+j)z^{-1}\right]\left[1-(1-j)z^{-1}\right]\left[1-(\tfrac{1}{2}+\tfrac{1}{2}j)z^{-1}\right]\left[1-(\tfrac{1}{2}-\tfrac{1}{2}j)z^{-1}\right]$$

$$= C_0\left[1-(1-j)z^{-1}-(1+j)z^{-1}+2z^{-2}\right]\left[1-(\tfrac{1}{2}-\tfrac{1}{2}j)z^{-1}-(\tfrac{1}{2}+\tfrac{1}{2}j)z^{-1}+\tfrac{1}{2}z^{-2}\right]$$

$$= C_0\left[1-2z^{-1}+2z^{-2}\right]\left[1-z^{-1}+\tfrac{1}{2}z^{-2}\right]$$

$$= C_0\Big[1 - z^{-1} + \tfrac{1}{2}z^{-2}$$
$$- 2z^{-1} + 2z^{-2} - z^{-3}$$
$$+ 2z^{-2} - 2z^{-3} + z^{-4}\Big]$$

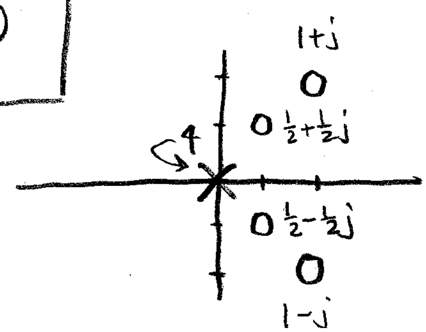
$$= C_0\left[1-3z^{-1}+\tfrac{9}{2}z^{-2}-3z^{-3}+z^{-4}\right]$$

@ $z=1$, $H(1) = C_0\left[1-3+\frac{9}{2}-3+1\right] = \frac{1}{2}C_0 = \frac{1}{2} \Rightarrow C_0 = 1.$

$$\boxed{H(z) = 1-3z^{-1}+\tfrac{9}{2}z^{-2}-3z^{-3}+z^{-4}, \quad |z|>0}$$

Also, $H(z) = \dfrac{1-3z+\frac{9}{2}z^2-3z^3+z^4}{z^4}$

$\Rightarrow$ Fourth-order pole @ $z=0$



Problem 2, cont...

(b) **8/6 pts**. Find the impulse response $h[n]$. Is H a Type 1, Type 2, Type 3, or Type 4 linear phase FIR filter?

Directly from $H(z)$, we have

$$\boxed{h[n] = \delta[n] - 3\delta[n-1] + \frac{9}{2}\delta[n-2] - 3\delta[n-3] + \delta[n-4]}$$

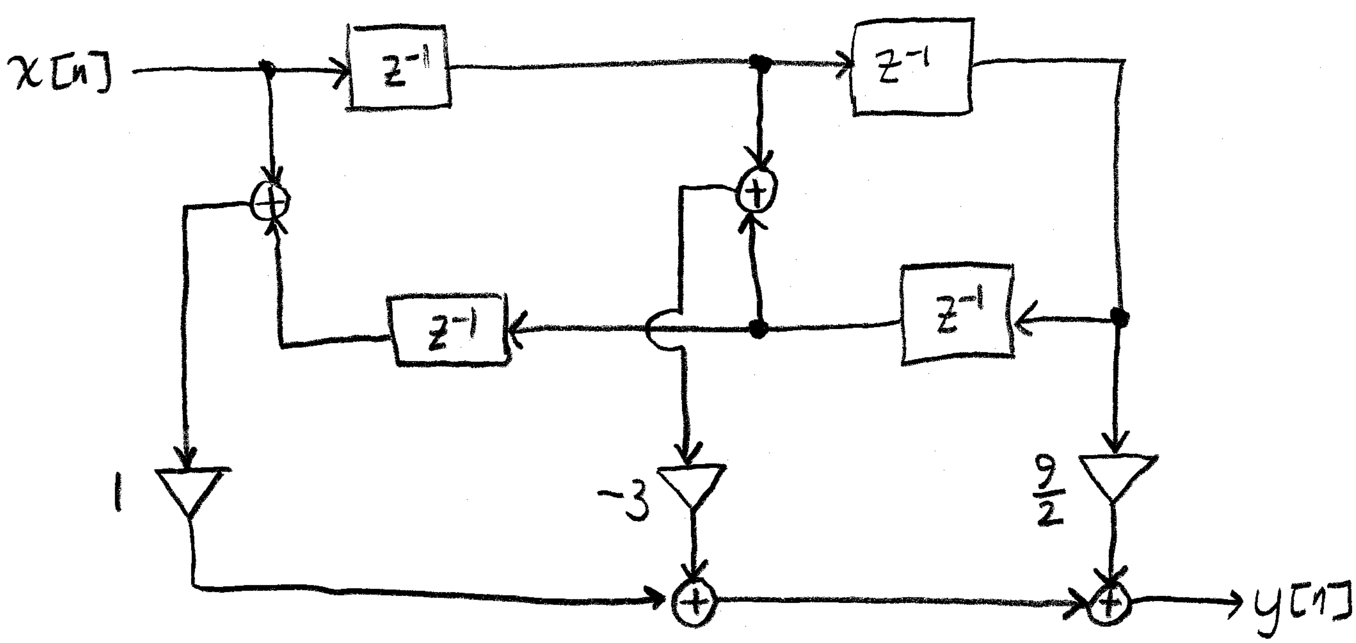
$$= [\,1 \quad -3 \quad \tfrac{9}{2} \quad -3 \quad 1\,]$$

$h[0] = h[4] = 1$
$h[1] = h[3] = -3$
$h[2] = 9/2$

$N = 4$
length $= 5$
Even Symmetry

$\boxed{\text{TYPE-1 FIR}}$

(c) **8/7 pts**. Give a linear-phase FIR structure (block diagram) for the filter H using the minimum number of multipliers.



3. **25/20 pts.** The causal IIR digital filter G has transfer function

$$G(z) = \frac{(1-2z^{-1})(1+2z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{3}{2}z^{-1}\right)}, \qquad |z| > \frac{3}{2}.$$

Note that this filter is undesirable for implementation because $G(z)$ is unstable and has maximum phase.

Design a new causal IIR filter H such that (1) H and G have the same magnitude response, *e.g.*, $|H(e^{j\omega})| = |G(e^{j\omega})| \ \forall\, \omega \in \mathbb{R}$, (2) H is causal and stable, and (3) $H(z)$ has minimum phase.

(a) **17/13 pts.** Find the transfer function $H(z)$.

We need to reflect the two bad zeros at ± 2 and the bad pole at $3/2$ inside the unit circle.

$$G(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} \frac{1}{1-\frac{3}{2}z^{-1}} (1-2z^{-1})(1+2z^{-1})$$

$$= \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} \frac{1}{(1-\frac{3}{2}z^{-1})} \frac{z^{-1}-\frac{3}{2}}{z^{-1}-\frac{3}{2}} (1-2z^{-1}) \frac{z^{-1}-2}{z^{-1}-2} (1+2z^{-1}) \frac{z^{-1}+2}{z^{-1}+2}$$

$$= \underbrace{\frac{(z^{-1}-2)(z^{-1}+2)}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})(z^{-1}-\frac{3}{2})}}_{H(z)} \cdot \underbrace{\frac{(z^{-1}-\frac{3}{2})(1-2z^{-1})(1+2z^{-1})}{(1-\frac{3}{2}z^{-1})(z^{-1}-2)(z^{-1}+2)}}_{\text{All Pass}}$$

$$H(z) = \frac{(-2)(1-\frac{1}{2}z^{-1})(2)\cancel{(1+\frac{1}{2}z^{-1})}}{(-\frac{3}{2})(1-\frac{2}{3}z^{-1})(1-\frac{1}{3}z^{-1})\cancel{(1+\frac{1}{2}z^{-1})}} = \frac{(-2)(2)(-\frac{2}{3})(1-\frac{1}{2}z^{-1})}{(1-\frac{2}{3}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$\boxed{H(z) = \frac{8}{3} \frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}+\frac{2}{9}z^{-2}}, \ |z| > \frac{2}{3}} = \frac{\frac{8}{3}-\frac{4}{3}z^{-1}}{1-z^{-1}+\frac{2}{9}z^{-2}}$$

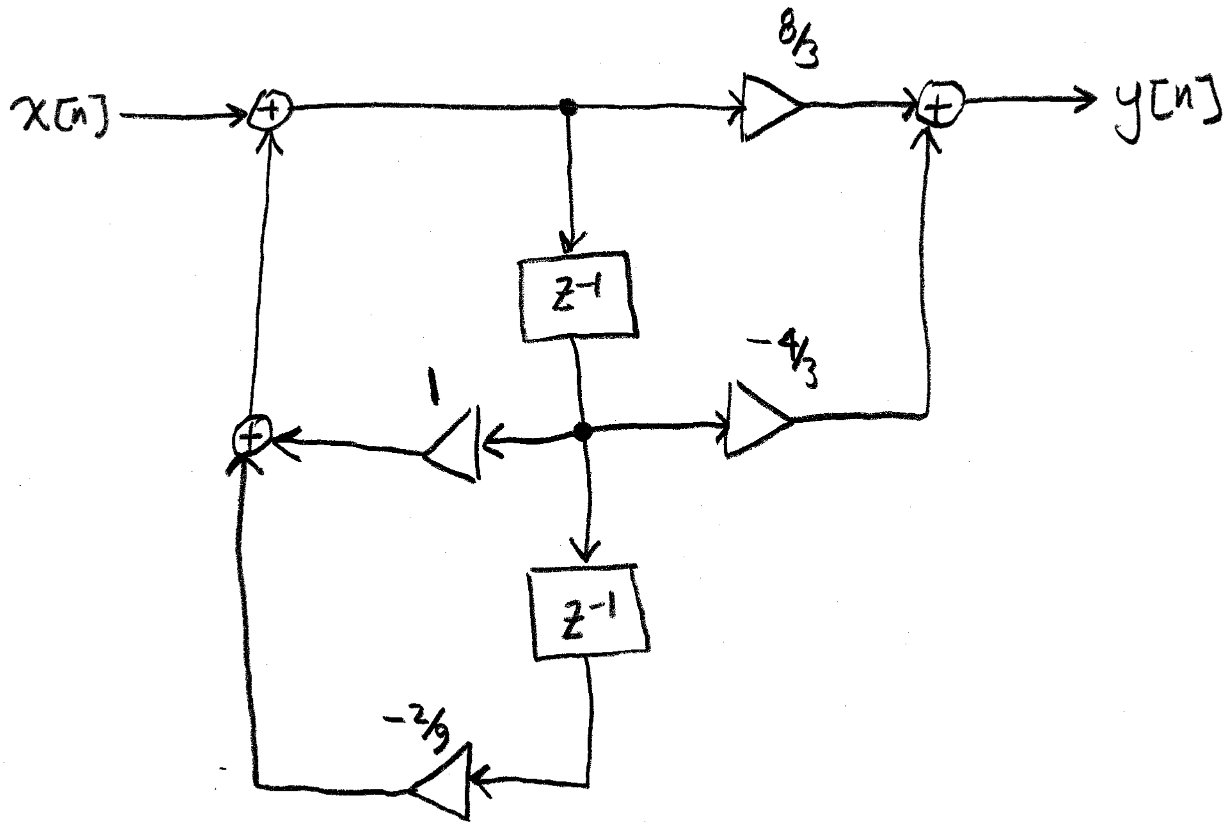
Problem 3, cont...

(b) **8/7 pts.** Give a direct-form II structure for $H(z)$.

Book (8.23) on p. 427:

$p_0 = \frac{8}{3}$ $p_1 = -\frac{4}{3}$ $d_1 = -1$ $d_2 = \frac{2}{9}$

Book Fig. 8.14 on p. 429:



4. **25/20 pts.** A digital FIR filter H has input $x[n]$ and output $y[n]$ related by

$$y[n] = x[n] - 2x[n-1] - 3x[n-2] + 4x[n-3] + 5x[n-4] - 6x[n-5].$$

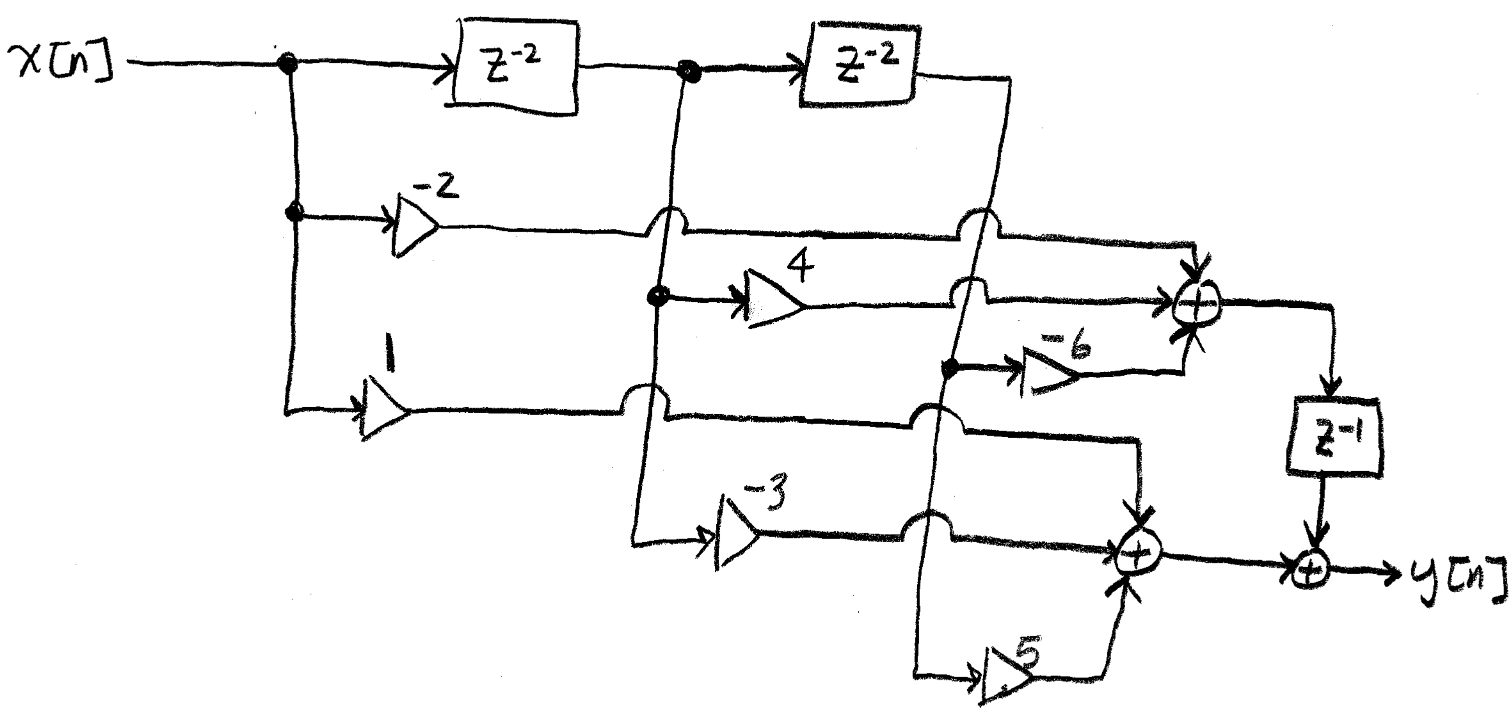
Find $H(z)$ and give a canonical direct polyphase structure using two phases.

z: $Y(z) = X(z)\left[1 - 2z^{-1} - 3z^{-2} + 4z^{-3} + 5z^{-4} - 6z^{-5}\right]$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = 1 - 2z^{-1} - 3z^{-2} + 4z^{-3} + 5z^{-4} - 6z^{-5}, \quad |z| > 0}$$

$$H(z) = (1 - 3z^{-2} + 5z^{-4}) + (-2z^{-1} + 4z^{-3} - 6z^{-5})$$
$$= \underbrace{(1 - 3z^{-2} + 5z^{-4})}_{E_0(z^2)} + z^{-1}\underbrace{(-2 + 4z^{-2} - 6z^{-4})}_{E_1(z^2)}$$

Book Figs. 8.7, 8.8:



5. **25/20 pts.** Design an **analog** Type II Chebyshev low pass filter to meet the following **analog** specification:

passband edge freq.	$\Omega_p = \pi$ rad/sec
stopband edge freq.	$\Omega_s = 2\pi$ rad/sec
stopband equiripple	$1/A = 1/\pi$
max. passband attenuation	$1/\sqrt{1+\varepsilon^2} = 0.9$

$A = \pi$

$\frac{1}{(.9)^2} = 1 + \varepsilon^2$

$\varepsilon^2 = \frac{1}{0.81} - 1 = 19/81$

$\varepsilon = \sqrt{\frac{19}{81}} = 0.484322$

Give the analog filter transfer function $H_a(s)$.

Hint: The design formulas for the analog Type II Chebyshev filter are given on page 869 of the text and pages 5-16B and 5-16C of the notes file `ECE5213NotesAnalogFilterDesignCh04.pdf`.

$$(A.17):\quad N = \left\lceil \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\cosh^{-1} 6.14919}{\cosh^{-1} 2} \right\rceil = \left\lceil \frac{2.50279}{1.31696} \right\rceil$$

$$= \lceil 1.90043 \rceil = 2.$$

$$(A.22):\quad z_1 = \frac{j2\pi}{\cos\left[\frac{(2-1)\pi}{4}\right]} = \frac{j2\pi}{\cos\frac{\pi}{4}} = j\frac{2\pi}{0.707107} = j8.88577$$

$$z_2 = j\frac{2\pi}{\cos\frac{3\pi}{4}} = j\frac{2\pi}{-0.707107} = -j8.88577$$

$$(A.24c):\quad \gamma = \left[A + \sqrt{A^2-1}\right]^{1/2} = \sqrt{\pi + \sqrt{\pi^2-1}} = 2.47382$$

$$\xi = \frac{\gamma^2+1}{2\gamma} = 1.43903$$

$$\zeta = \frac{\gamma^2-1}{2\gamma} = 1.03479$$

(A.24b):

$$\alpha_1 = -\Omega_p \zeta \sin\frac{\pi}{4} = -2.29873$$

$$\alpha_2 = -\Omega_p \zeta \sin\frac{3\pi}{4} = -2.29873$$

$$\beta_1 = \Omega_p \xi \cos\frac{\pi}{4} = 3.19671$$

$$\beta_2 = \Omega_p \xi \cos\frac{3\pi}{4} = -3.19671$$

$$(A.24a):\quad \sigma_1 = \frac{\Omega_s \alpha_1}{\alpha_1^2+\beta_1^2} = -0.931641$$

$$\Omega_1 = \frac{-\Omega_s \beta_1}{\alpha_1^2+\beta_1^2} = -1.29558$$

$$\sigma_2 = \frac{\Omega_s \alpha_2}{\alpha_2^2+\beta_2^2} = -0.931641$$

$$\Omega_2 = \frac{-\Omega_s \beta_2}{\alpha_2^2+\beta_2^2} = 1.29558$$

(A.23) : $p_1 = \sigma_1 + j\Omega_1 = -0.931641 - j1.29558$

$p_2 = \sigma_2 + j\Omega_2 = -0.931641 + j1.29558$

(A.21): $H_a(s) = C_0 \dfrac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)}$

$$H_a(0) = C_0 \frac{(-z_1)(-z_2)}{(-p_1)(-p_2)} = C_0 \frac{z_1 z_2}{p_1 p_2} = 1$$

$$\Rightarrow C_0 = \frac{p_1 p_2}{z_1 z_2} = 0.0322515$$

$$H_a(s) = \frac{p_1 p_2}{z_1 z_2} \frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)} = \frac{p_1 p_2}{z_1 z_2} \frac{s^2 - (z_1+z_2)s + z_1 z_2}{s^2 - (p_1+p_2)s + p_1 p_2}$$

$$= \frac{\frac{|p_1|^2}{|z_1|^2}\left[s^2 - \cancel{2\mathrm{Re}\{z_1\}}^{\,0}s + |z_1|^2\right]}{s^2 - 2\mathrm{Re}\{p_1\}s + |p_1|^2} = \frac{\frac{|p_1|^2}{|z_1|^2}s^2 - \cancel{\frac{2\mathrm{Re}\{z_1\}|p_1|^2}{|z_1|^2}}^{\,0}s + |p_1|^2}{s^2 - 2\mathrm{Re}\{p_1\}s + |p_1|^2}$$

$$\boxed{H_a(s) = \frac{0.0322515\,s^2 + 2.5465}{s^2 + 1.8633s + 2.5465}, \quad \mathrm{Re}[s] > -0.931641}$$