ECE 4213/5213
Test 2

Wedneday, November 28, 2012
4:30 PM - 5:45 PM

Fall 2012 — Name: SOLUTION

Dr. Havlicek — Student Num:__________

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

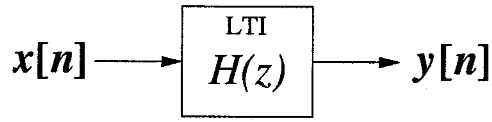
1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

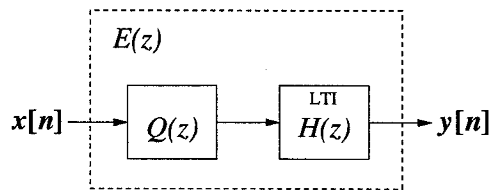
1. **25/20 pts.** Consider the non-ideal digital communications channel H shown below.



It is determined by experiment that this channel can be modeled as a **causal** IIR LTI system with transfer function

$$H(z) = \frac{\left(1 - \frac{3}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)\left(1 + \frac{5}{3}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)^2\left(1 - \frac{1}{4}z^{-1}\right)}.$$

Design a digital pre-equalizer $Q(z)$ to go in series with the channel as shown below



so that the overall *equalized* channel $E(z)$ is **allpass**.

It is required for your pre-equalizer $Q(z)$ to be both **causal** and **stable** and to have **minimum group delay**.

Because $H(z)$ has two zeros outside the unit circle, we can't take $Q(z) = 1/H(z)$ ... this would give $Q(z)$ two poles outside the unit circle so that $Q(z)$ could not be causal and stable. So we will find a new minimum phase $H'(z)$ that has the same magnitude as $H(z)$. Then $Q(z) = 1/H'(z)$.

→ since $H'(z)$ will be minimum phase, the resulting $Q(z)$ can be causal and stable.

→ since $H'(z)$ will have the same poles as $H(z)$, which are both inside the unit circle, the resulting $Q(z)$ will also be minimum phase, implying minimum group delay.

More Workspace for Problem 1...

$$H(z) = \frac{(1+\frac{1}{3}z^{-1})(1-\frac{3}{2}z^{-1})(1+\frac{5}{3}z^{-1})}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})}$$

$$= \frac{(1+\frac{1}{3}z^{-1})}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})}\,(1-\tfrac{3}{2}z^{-1})\,\frac{z^{-1}-\frac{3}{2}}{z^{-1}-\frac{3}{2}}\,(1+\tfrac{5}{3}z^{-1})\,\frac{z^{-1}+5/3}{z^{-1}+5/3}$$

$$= \underbrace{\frac{(1+\frac{1}{3}z^{-1})(z^{-1}-\frac{3}{2})(z^{-1}+\frac{5}{3})}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})}}_{H'(z)} \cdot \underbrace{\frac{(1-\frac{3}{2}z^{-1})(1+\frac{5}{3}z^{-1})}{(z^{-1}-3/2)(z^{-1}+5/3)}}_{\text{all pass}}$$

$$H'(z) = \frac{(1+\frac{1}{3}z^{-1})(\frac{-3}{2})(1-\frac{2}{3}z^{-1})(5/3)(1+3/5\,z^{-1})}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})}$$

$$= \frac{-5/2(1+\frac{1}{3}z^{-1})(1-2/3\,z^{-1})(1+3/5\,z^{-1})}{(1-1/2\,z^{-1})^2(1-\frac{1}{4}z^{-1})}$$

$$\boxed{Q(z) = \frac{1}{H'(z)} = \frac{(-\frac{2}{5})(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{4}z^{-1})}{(1+\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})(1+\frac{3}{5}z^{-1})}}$$

2. **25/20 pts.** An FIR linear phase filter has a **real-valued** impulse response $h[n]$ that is nonzero only for $0 \le n \le 6$. In addition, it is known that $h[0] = 1$.

The transfer function $H(z)$ has a complex zero at $z = 0.4e^{-j\pi/3}$ and a real zero at $z = 3$. **Note:** these are not the *only* zeros.

Find $H(z)$. Length $h[n] = 7 \rightarrow$ order $= N = 6$.

$\rightarrow H(z)$ has six zeros.

$h[n]$ real $\longrightarrow$ any complex zeros occur in conjugate pairs

linear phase FIR $\rightarrow$ having a zero at $z = z_k$ implies there is also a zero at $z = 1/z_k$.

We are given $z_1 = \frac{4}{10} e^{-j\pi/3}$. Since $h[n]$ is real, $z_2 = \frac{4}{10} e^{j\pi/3}$.

Since H is linear phase FIR, $z_3 = \frac{10}{4} e^{j\pi/3}$ and $z_4 = \frac{10}{4} e^{-j\pi/3}$

We are also given $z_5 = 3$. Since linear phase FIR, $z_6 = 1/3$.

$$H(z) = \sum_{n=0}^{6} h[n] z^{-1} = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6}. \quad (*)$$

Also, $$H(z) = A \prod_{k=1}^{6} (1 - z_k z^{-1})$$

$$= A\left(1 - \tfrac{2}{5} e^{-j\pi/3} z^{-1}\right)\left(1 - \tfrac{2}{5} e^{j\pi/3}\right)\left(1 - \tfrac{5}{2} e^{j\pi/3} z^{-1}\right)\left(1 - \tfrac{5}{2} e^{-j\pi/3} z^{-1}\right) \times \left(1 - 3z^{-1}\right)\left(1 - \tfrac{1}{3} z^{-1}\right)$$

$$= A + A(\text{terms involving } z^{-1}) \quad (**).$$

$(*)$ and $(**)$ imply $A = h[0] = 1$.

$$\boxed{H(z) = \left(1 - \tfrac{2}{5} e^{-j\pi/3} z^{-1}\right)\left(1 - \tfrac{2}{5} e^{j\pi/3} z^{-1}\right)\left(1 - \tfrac{5}{2} e^{j\pi/3} z^{-1}\right)\left(1 - \tfrac{5}{2} e^{-j\pi/3} z^{-1}\right)\left(1 - 3z^{-1}\right)\left(1 - \tfrac{1}{3} z^{-1}\right)}$$

3. **25/20 pts.** Design an **analog** Butterworth low pass filter to meet the following **analog** specification:

passband edge freq.	$\Omega_p = 2500\pi$ rad/sec
stopband edge freq.	$\Omega_s = 7000\pi$ rad/sec
min. stopband attenuation	$1/A = 0.063$
max. passband attenuation	$1/\sqrt{1+\varepsilon^2} = 0.95$

$$A = \frac{1}{0.063} = 15.8730$$

$$1+\varepsilon^2 = \frac{1}{(.95)^2}$$

$$\varepsilon^2 = \frac{1}{(.95)^2} - 1$$

$$\varepsilon^2 = 0.108033$$

Give the analog filter transfer function $H_a(s)$.

Hint: The design formulas for the analog Butterworth filter are given on pages 5-13 and 5-14 of the notes file `ECE5213NotesAnalogFilterDesignCh04.pdf` and in Appendix A.2 on page 865 of the text.

$$N = \left\lceil \frac{\frac{1}{2}\log_{10}\left[(A^2-1)/\varepsilon^2\right]}{\log_{10} \Omega_s/\Omega_p} \right\rceil = \left\lceil \frac{1.68302}{0.447158} \right\rceil = \lceil 3.76381 \rceil = 4$$

$$\frac{1}{A^2} = \frac{1}{1+(\Omega_s/\Omega_c)^8} \Rightarrow A^2 = 1 + \frac{\Omega_s^8}{\Omega_c^8} \Rightarrow A^2 - 1 = \frac{\Omega_s^8}{\Omega_c^8} \Rightarrow \Omega_c^8 = \frac{\Omega_s^8}{A^2-1}$$

$$\Omega_c = \frac{\Omega_s}{(A^2-1)^{1/8}} = \frac{7000\pi}{1.99503} = 3508.72\pi = 11{,}022.98$$

poles: $p_\ell = \Omega_c e^{j[\pi(N+2\ell-1)/2N]}, \quad 1 \le \ell \le 4$

$\ell = 1: \quad p_1 = 11{,}022.98\, e^{j5\pi/8}$

$\ell = 2: \quad p_2 = 11{,}022.98\, e^{j7\pi/8}$

$\ell = 3: \quad p_3 = 11{,}022.98\, e^{j9\pi/8}$

$\ell = 4: \quad p_4 = 11{,}022.98\, e^{j11\pi/8}$

$\longrightarrow$

More Workspace for Problem 3...

$$H_a(s) = \frac{\Omega_c^4}{\prod_{\ell=1}^{4} (s-p_\ell)}$$

$$\boxed{H_a(s) = \frac{14.763719 \times 10^{15}}{(s-11{,}022.98e^{j5\pi/8})(s-11{,}022.98e^{j7\pi/8})(s-11{,}022.98e^{j9\pi/8})(s-11{,}022.98e^{j11\pi/8})}}$$

4. **25/20 pts.** Use the window design method with an appropriate fixed window from Table 10.2 (p. 540 of the text) to design a causal lowpass FIR digital filter that meets the following design specification:

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.51\pi + 0.6\pi}{2}$$

$$\omega_c = 0.555\pi$$

Passband Edge Freq.	$\omega_p = 0.51\pi$ rad/sample
Stopband Edge Freq.	$\omega_s = 0.60\pi$ rad/sample
Max. Passband Ripple	$\delta_p = 0.001$
Max. Stopband Ripple	$\delta_s = 0.001$

From (9.4):

$$\alpha_s = -20\log_{10}\delta_s = -20\log_{10} 0.001$$

$$\alpha_s = 60\text{ dB}.$$

Give the filter impulse response $h[n]$.

From Table 10.2, only the Blackman window can provide enough Stop band attenuation, $\Rightarrow$ Use Blackman.

Transition Bandwidth: $\Delta\omega = \omega_s - \omega_p = 0.6\pi - 0.51\pi = 0.09\pi$

Table 10.2: $\Delta\omega = 0.09\pi = \dfrac{5.56\pi}{M}$

$$M = \left\lceil \frac{5.56\pi}{0.09\pi} \right\rceil = \lceil 61.7778 \rceil = 62$$

Length $= 2M+1 = 125$

Order $= 2M = 124$

(10.14): $h_{LP}[n] = \dfrac{\sin\omega_c n}{\pi n} = \dfrac{\sin(0.555\pi n)}{\pi n}$

(10.35): $w[n] = 0.42 + 0.5\cos\left(\frac{\pi n}{M}\right) + 0.08\cos\left(\frac{2\pi n}{M}\right), \quad -62 \le n \le 62$

$= 0.42 + 0.5\cos\left(\frac{\pi}{62}n\right) + 0.08\cos\left(\frac{\pi}{31}n\right), \quad -62 \le n \le 62$

$$w[n]h_{LP}[n] = \left[0.42 + 0.5\cos\left(\frac{\pi}{62}n\right) + 0.08\cos\left(\frac{\pi}{31}n\right)\right]\frac{\sin(0.555\pi n)}{\pi n}, \quad -62 \le n \le 62$$

Shift to make causal:

$$\boxed{h[n] = \left\{0.42 + 0.5\cos\left[\frac{\pi}{62}(n-62)\right] + 0.08\cos\left[\frac{\pi}{31}(n-62)\right]\right\}\frac{\sin[0.555\pi(n-62)]}{\pi(n-62)}, \quad 0 \le n \le 124}$$

5. **25/20 pts**. Use the bilinear transform with $T = 2$ to design a Chebyshev Type 1 digital lowpass filter that meets the following specifications:

passband edge freq.	$\omega_p = 0.2\pi$ rad/sample
stopband edge freq.	$\omega_s = 0.8\pi$ rad/sample
max. passband ripple	$\frac{1}{\sqrt{1+\epsilon^2}} = 0.8$
min. stopband atten.	$1/A = 0.2$

Give the digital filter transfer function $H(z)$.

(a) **4/3 pts**. Use the bilinear transform with $T = 2$ to find the critical analog frequencies Ω_p and Ω_s by prewarping the digital frequencies ω_p and ω_s.

(9.20): $\Omega_p = \tan \frac{\omega_p}{2} = \tan(0.1\pi) = 0.324920$

$\Omega_s = \tan \frac{\omega_s}{2} = \tan(0.4\pi) = 3.07768$

(b) **7/6 pts**. Find the required filter order N.

$\frac{1}{A} = \frac{2}{10}$; $A = \frac{10}{2} = 5$; $A^2 = 25$

$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{8}{10}$; $1+\varepsilon^2 = \frac{100}{64}$; $\varepsilon^2 = \frac{100}{64} - 1 = \frac{100-64}{64} = \frac{36}{64}$

$\varepsilon = \frac{6}{8} = \frac{3}{4}$

(A.17): $N = \left\lceil \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\cosh^{-1}\left(\frac{4\sqrt{24}}{3}\right)}{\cosh^{-1}(9.47214)} \right\rceil$

$= \left\lceil \frac{\cosh^{-1}(6.53197)}{\cosh^{-1}(9.47214)} \right\rceil = \left\lceil \frac{2.56394}{2.93870} \right\rceil$

$= \lceil 0.872475 \rceil = 1$

$$\boxed{N = 1}$$

Problem 5, cont...

(c) **7/6 pts.** Find the poles and give an explicit expression for $H_a(s)$.

Hint: the design formulas are given on pages 5-16 and 5-16B of the course notes in the file `ECE5213NotesAnalogFilterDesignCh04.pdf` and on p. 868 of the text in equations (A.18) through (A.19b).

(A.19b): $\gamma = \left(\dfrac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} \right)^{\frac{1}{N}} = \dfrac{1+\frac{10}{8}}{3/4} = \dfrac{1+5/4}{3/4} = \dfrac{4}{3} \cdot \dfrac{9}{4} = 3$

$\zeta = \dfrac{\gamma^2+1}{2\gamma} = \dfrac{10}{6} = \dfrac{5}{3}$; $\xi = \dfrac{\gamma^2-1}{2\gamma} = \dfrac{8}{6} = \dfrac{4}{3}$

(A.19a): $\sigma_1 = -\Omega_p \xi \sin\frac{\pi}{2} = -\Omega_p \xi = -\frac{4}{3}\Omega_p = -0.433226$

$\Omega_1 = \Omega_p \zeta \cos\frac{\pi}{2} = 0$

(A.18): $p_1 = \sigma_1 + j\Omega_1 = -0.433226$

Notes p. 5-16B: $H_a(s) = C_0 \prod_{\ell=1}^{N} \dfrac{-p_\ell}{s-p_\ell}$

N odd: $C_0 = 1$

$$\boxed{H_a(s) = \frac{0.433226}{s+0.433226}}$$

Problem 5, cont...

(d) **7/5 pts**. Use the bilinear transform to find $H(z)$.

(9.15): $$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{-p_1}{s-p_1}\Bigg|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{-p_1}{\frac{1-z^{-1}}{1+z^{-1}} - p_1} \cdot \frac{1+z^{-1}}{1+z^{-1}} = \frac{-p_1 - p_1 z^{-1}}{1 - z^{-1} - (1+z^{-1})p_1}$$

$$= \frac{-p_1 - p_1 z^{-1}}{1 - z^{-1} - p_1 - p_1 z^{-1}} = \frac{-p_1 - p_1 z^{-1}}{(1-p_1) - (1+p_1) z^{-1}}$$

$$= \frac{\frac{-p_1}{1-p_1} - \frac{p_1}{1-p_1} z^{-1}}{1 - \frac{1+p_1}{1-p_1} z^{-1}} = \frac{0.302273 + 0.302273 z^{-1}}{1 - 0.395453 z^{-1}}$$

$$\boxed{H(z) = \frac{0.302273 + 0.302273 z^{-1}}{1 - 0.395453 z^{-1}}, \quad |z| > 0.395453}$$