ECE 4213/5213
Test 2

Monday, November 24, 2014
4:30 PM - 5:45 PM

Fall 2014 Name: SOLUTION

Dr. Havlicek Student Num: __________

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work problem 5 and select any three out of problems 1-4. Each problem counts 25 points. Below, circle the numbers of the three problems you wish to have graded out of problems 1-4.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: __________ Date: __________

1. **25/20 pts**. Consider a causal FIR digital filter H with input $x[n]$ and output $y[n]$ related by

$$y[n] = -x[n] + 2x[n-1] - x[n-2].$$

The finite-length input signal is given by

$$\begin{aligned} x[n] &= [1 \quad 2 \quad -3 \quad -2 \quad -1] \\ &= \delta[n] + 2\delta[n-1] - 3\delta[n-2] - 2\delta[n-3] - \delta[n-4], \quad 0 \le n \le 4. \end{aligned}$$

Use the DFT to find the finite-length output signal $y[n]$.

NOTE: in this problem, you are being asked to use the DFT to implement *linear convolution.*

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = -1x[n-0] + 2x[n-1] - 1x[n-2]$$

$$\Rightarrow h[n] = -\delta[n] + 2\delta[n-1] - \delta[n-2]$$

$$= [-1 \quad 2 \quad -1]$$

$$N_1 = 3$$

$$x[n] = [1 \quad 2 \quad -3 \quad -2 \quad -1], \quad N_2 = 5$$

$$N = N_1 + N_2 - 1 = 3 + 5 - 1 = 7.$$

$$h'[n] = [-1 \quad 2 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$H'[k] = \sum_{n=0}^{6} h'[n] W_7^{nk} = -1 + 2W_7^k - W_7^{2k}$$

$$x'[n] = [1 \quad 2 \quad -3 \quad -2 \quad -1 \quad 0 \quad 0]$$

$$X'[k] = \sum_{n=0}^{6} x'[n] W_7^{nk} = 1 + 2W_7^k - 3W_7^{2k} - 2W_7^{3k} - W_7^{4k}$$

$\longrightarrow$

More Workspace for Problem 1...

$$Y[k] = H'[k]X'[k] = -X'[k] + 2W_7^k X'[k] - W_7^{2k} X'[k]$$

$$= -1 - 2W_7^k + 3W_7^{2k} + 2W_7^{3k} + W_7^{4k}$$
$$+ 2W_7^k + 4W_7^{2k} - 6W_7^{3k} - 4W_7^{4k} - 2W_7^{5k}$$
$$- W_7^{2k} - 2W_7^{3k} + 3W_7^{4k} + 2W_7^{5k} + W_7^{6k}$$

$$Y[k] = -1 + 0W_7^k + 6W_7^{2k} - 6W_7^{3k} + 0W_7^{4k} + 0W_7^{5k} + W_7^{6k}$$

$$= \sum_{n=0}^{6} y[n] W_7^{nk}$$

$$\Rightarrow y[n] = [-1 \quad 0 \quad 6 \quad -6 \quad 0 \quad 0 \quad 1]$$

$$y[n] = -\delta[n] + 6\delta[n-2] - 6\delta[n-3] + \delta[n-6], \quad 0 \le n \le 6.$$

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2. **25/20 pts.** Consider a discrete-time LTI system H_1 with input $x[n]$ and output $y[n]$ related by

$$y[n] + \frac{1}{2}y[n-1] - \frac{3}{16}y[n-2] = x[n] + \frac{5}{3}x[n-1] - \frac{2}{3}x[n-2].$$

The system H_1 is causal and BIBO stable, but it does not have minimum phase.

Give the I/O relation for a new causal, stable discrete-time system H_2 that has the same magnitude response as H_1 but also has minimum phase.

$$Y(z) + \tfrac{1}{2}z^{-1}Y(z) - \tfrac{3}{16}z^{-2}Y(z) = X(z) + \tfrac{5}{3}z^{-1}X(z) - \tfrac{2}{3}z^{-2}X(z)$$

$$Y(z)\left[1 + \tfrac{1}{2}z^{-1} - \tfrac{3}{16}z^{-2}\right] = X(z)\left[1 + \tfrac{5}{3}z^{-1} - \tfrac{2}{3}z^{-2}\right]$$

$$H_1(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{5}{3}z^{-1} - \frac{2}{3}z^{-2}}{1 + \frac{1}{2}z^{-1} - \frac{3}{16}z^{-2}} = \frac{(1+2z^{-1})(1-\frac{1}{3}z^{-1})}{(1+\frac{3}{4}z^{-1})(1-\frac{1}{4}z^{-1})}$$

→ Not minimum phase, since the zero @ $z=-2$ is outside the unit circle.

$$H_1(z) = \frac{(1-\frac{1}{3}z^{-1})}{(1+\frac{3}{4}z^{-1})(1-\frac{1}{4}z^{-1})}(1+2z^{-1})\underbrace{\frac{z^{-1}+2}{z^{-1}+2}}_{\text{"one"}}$$

$$= \underbrace{\frac{(1-\frac{1}{3}z^{-1})(z^{-1}+2)}{(1+\frac{3}{4}z^{-1})(1-\frac{1}{4}z^{-1})}}_{H_2(z)}\underbrace{\left[\frac{1+2z^{-1}}{z^{-1}+2}\right]}_{\text{All Pass}}$$

$$H_2(z) = \frac{2(1+\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}{(1+\frac{3}{4}z^{-1})(1-\frac{1}{4}z^{-1})}, \quad |z| > \frac{3}{4}.$$

poles: $z = -\frac{3}{4}, +\frac{1}{4}$
→ causal, stable ✓
zeros: $z = -\frac{1}{2}, +\frac{1}{3}$
→ minimum phase ✓

$$= \frac{2(1+\frac{1}{2}z^{-1}-\frac{1}{3}z^{-1}-\frac{1}{6}z^{-2})}{1+\frac{3}{4}z^{-1}-\frac{1}{4}z^{-1}-\frac{3}{16}z^{-2}} = \frac{2+\frac{1}{3}z^{-1}-\frac{1}{3}z^{-2}}{1+\frac{1}{2}z^{-1}-\frac{3}{16}z^{-2}} = \frac{Y(z)}{X(z)}$$

$$Y(z) + \tfrac{1}{2}z^{-1}Y(z) - \tfrac{3}{16}z^{-2}Y(z) = 2X(z) + \tfrac{1}{3}z^{-1}X(z) - \tfrac{1}{3}z^{-2}X(z)$$

$$// \; y[n] + \tfrac{1}{2}y[n-1] - \tfrac{3}{16}y[n-2] = 2x[n] + \tfrac{1}{3}x[n-1] - \tfrac{1}{3}x[n-2] \; //$$

3. **25/20 pts.** H is a causal, stable type III linear phase FIR digital filter with order $N = 6$. The impulse response $h[n]$ is real and has length $N + 1 = 7$.

The value of $H(z)$ at $z = \frac{1}{2}$ is given by $H(\frac{1}{2}) = -39$.

$H(z)$ has a complex zero at $z = \frac{1}{2}e^{j\frac{\pi}{3}}$.

Find the transfer function $H(z)$ and the impulse response $h[n]$.

Order $N=6 \Rightarrow$ There are 6 zeros

(because H is FIR, there are also 6 poles at $z=0$, but when $H(z)$ is expressed as a function of z^{-1} the denominator is 1).

So $H(z) = K \prod_{m=1}^{6} (1 - z_m z^{-1})$, where K is a constant and z_m are the zeros.

- Because H is a type III linear phase FIR filter, there must be zeros at $z=1$ ($\omega=0$) and $z=-1$ ($\omega = \pm\pi$) (see notes, p. 7.49).
- It is given that there is a zero at $z = \frac{1}{2}e^{j\pi/3}$.
- Because $h[n]$ is real, all complex zeros must occur in conjugate pairs. So there must be another zero at $z = \frac{1}{2}e^{-j\pi/3}$.
- Because H is a linear phase FIR filter, all zeros that are off the unit circle must occur in mirror image pairs. So there must be another zero at $z = 1/(\frac{1}{2}e^{j\pi/3}) = 2e^{-j\pi/3}$
- Again because $h[n]$ is real, there must be another zero at $z = 2e^{j\pi/3}$.

$\Rightarrow$ The six zeros are at $z = 1, -1, \frac{1}{2}e^{j\pi/3}, \frac{1}{2}e^{-j\pi/3}, 2e^{j\pi/3},$ and $2e^{-j\pi/3}$. $\rightarrow$

More Workspace for Problem 3...

$$H(z) = K(1-z^{-1})(1+z^{-1})(1-\tfrac{1}{2}e^{j\pi/3}z^{-1})(1-\tfrac{1}{2}e^{-j\pi/3}z^{-1})$$
$$\cdot(1-2e^{j\pi/3}z^{-1})(1-2e^{-j\pi/3}z^{-1})$$

b/c H is causal: ROC: $|z|>0$

$$= K(1-z^{-2})\left[1-\tfrac{1}{2}(e^{j\pi/3}+e^{-j\pi/3})z^{-1}+\tfrac{1}{4}z^{-2}\right]$$
$$\cdot\left[1-2(e^{j\pi/3}+e^{-j\pi/3})z^{-1}+4z^{-2}\right]$$

$\boxed{\cos\frac{\pi}{3}=\frac{1}{2}}$

$$= K(1-z^{-2})\left[1-\tfrac{1}{2}\cdot 2\cos\tfrac{\pi}{3}z^{-1}+\tfrac{1}{4}z^{-2}\right]\left[1-2\cdot 2\cos\tfrac{\pi}{3}z^{-1}+4z^{-2}\right]$$
$$= K(1-z^{-2})\left[1-\tfrac{1}{2}z^{-1}+\tfrac{1}{4}z^{-2}\right]\left[1-2z^{-1}+4z^{-2}\right]$$
$$= K(1-z^{2})\left(1-2z^{-1}+4z^{-2}-\tfrac{1}{2}z^{-1}+z^{-2}-2z^{-3}+\tfrac{1}{4}z^{-2}-\tfrac{1}{2}z^{-3}+z^{-4}\right)$$
$$= K(1-z^{-2})\left(1-\tfrac{5}{2}z^{-1}+\tfrac{21}{4}z^{-2}-\tfrac{5}{2}z^{-3}+z^{-4}\right)$$
$$= K\Big(1-\tfrac{5}{2}z^{-1}+\tfrac{21}{4}z^{-2}-\tfrac{5}{2}z^{-3}+z^{-4}$$
$$-z^{-2}+\tfrac{5}{2}z^{-3}-\tfrac{21}{4}z^{-4}+\tfrac{5}{2}z^{-5}-z^{-6}\Big)$$

$$H(z) = K\left(1-\tfrac{5}{2}z^{-1}+\tfrac{17}{4}z^{-2}+0z^{-3}-\tfrac{17}{4}z^{-4}+\tfrac{5}{2}z^{-5}-z^{-6}\right) //$$

$$H(\tfrac{1}{2}) = -39 = K\left(1-\tfrac{5}{2}\cdot 2+\tfrac{17}{4}\cdot 4+0-\tfrac{17}{4}\cdot 16+\tfrac{5}{2}\cdot 32-64\right)$$
$$= K(1-5+17-17\cdot 4+5\cdot 16-64)$$
$$= K(13-68+80-64) = K(-39) \Rightarrow \underline{K=1}.$$

$$\underline{\underline{H(z) = 1-\tfrac{5}{2}z^{-1}+\tfrac{17}{4}z^{-2}+0z^{-3}-\tfrac{17}{4}z^{-4}+\tfrac{5}{2}z^{-5}-z^{-6},\quad |z|>0}}$$

$$\underline{\underline{h[n] = \delta[n]-\tfrac{5}{2}\delta[n-1]+\tfrac{17}{4}\delta[n-2]-\tfrac{17}{4}\delta[n-4]+\tfrac{5}{2}\delta[n-5]-\delta[n-6]}}$$

Symmetry Check: $h[n] = \left[1\ \ -\tfrac{5}{2}\ \ \tfrac{17}{4}\ \ 0\ \ -\tfrac{17}{4}\ \ \tfrac{5}{2}\ \ -1\right]$ ✓

Type III linear phase FIR

The numbers in boxes remind me which calculator register I stored things in $\boxed{00}$

4. **25/20 pts.** Design an analog Type 1 Chebyshev low pass filter to meet the following analog design specification:

passband edge freq.	$\Omega_p = 1000\pi$ rad/sec
stopband edge freq.	$\Omega_s = 5000\pi$ rad/sec
min. stopband attenuation	$1/A = 0.063$
max. passband attenuation	$1/\sqrt{1+\varepsilon^2} = 0.95$

$$A = \frac{1}{0.063} = 15.8730 \quad \boxed{00}$$

$$\frac{1}{.95} = \sqrt{1+\varepsilon^2}$$

$$1+\varepsilon^2 = \frac{1}{(.95)^2}$$

Give the analog transfer function $H_a(s)$.

$$\varepsilon = \sqrt{\frac{1}{(.95)^2} - 1} = 0.328684 \quad \boxed{01}$$

Hint: The design formulas for the analog Type 1 Chebyshev filter are given on pages A-7 and A-8 of the notes file `ECE5213NotesAnalogFilterDesign.pdf` and in Appendix A.3 of the text on pages 867 and 868.

$$\text{(A.17)} \quad N = \left\lceil \frac{\cosh^{-1}\left(\sqrt{A^2-1}/\varepsilon\right)}{\cosh^{-1}\left(\Omega_s/\Omega_p\right)} \right\rceil = \left\lceil \frac{\cosh^{-1}(\overset{\boxed{02}}{48.1967})}{\cosh^{-1}(5)} \right\rceil = \left\lceil \frac{\overset{\boxed{03}}{4.56833}}{2.29243} \right\rceil$$

$$= \lceil 1.99279 \rceil = 2.$$

$$\text{(A.19b)} \quad \gamma = \sqrt{\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}} = 2.49900 \quad \boxed{04}$$

$$\zeta = \frac{\gamma^2+1}{2\gamma} = 1.44958 \quad \boxed{05} \qquad \xi = \frac{\gamma^2-1}{2\gamma} = 1.04942 \quad \boxed{06}$$

$$\text{(A.19a)} \quad \sigma_1 = -\Omega_p \xi \sin\frac{\pi}{4} = -1000\pi\xi\sin\frac{\pi}{4} = -2331.22 \quad \boxed{07}$$

$$\sigma_2 = -\Omega_p \xi \sin\frac{3\pi}{4} = -1000\pi\xi\sin\frac{3\pi}{4} = -2331.22 = \sigma_1$$

$$\Omega_1 = \Omega_p \zeta \cos\frac{\pi}{4} = 1000\pi\zeta\cos\frac{\pi}{4} = 3220.16 \quad \boxed{08}$$

$$\Omega_2 = \Omega_p \zeta \cos\frac{3\pi}{4} = 1000\pi\zeta\cos\frac{3\pi}{4} = -3220.16 = -\Omega_1$$

$$\text{(A.18)} \quad p_1 = \sigma_1 + j\Omega_1 = -2331.22 + j3220.16$$

$$p_2 = \sigma_2 + j\Omega_2 = -2331.22 - j3220.16 = p_1^*$$

More Workspace for Problem 4...

Notes p. A-8: for N even, $C_0 = \frac{1}{\sqrt{1+\varepsilon^2}} = 0.95$

Notes p. A-8: $H_a(s) = C_0 \frac{-p_1}{s-p_1} \cdot \frac{-p_2}{s-p_2}$

$$= \frac{C_0\, p_1 p_1^*}{(s-p_1)(s-p_1^*)} = \frac{C_0\, p_1 p_1^*}{s^2 - (p_1+p_1^*)s + p_1 p_1^*}$$

$$= \frac{C_0 |p_1|^2}{s^2 - 2\mathrm{Re}[p_1]s + |p_1|^2}$$

$$= \frac{C_0(\sigma_1^2+\Omega_1^2)}{s^2 - 2\sigma_1 s + (\sigma_1^2+\Omega_1^2)}$$

$$= \frac{(0.95)\left[(-2331.22)^2 + (3220.16)^2\right]}{s^2 - 2(-2331.22)s + \left[(-2331.22)^2 + (3220.16)^2\right]}$$

$$\boxed{H_a(s) = \frac{15.0138 \times 10^6}{s^2 + 4662.45s + 15.8040 \times 10^6}}$$

Numbers in boxes like $\boxed{00}$ remind me which calculator register I stored things in.

5. **25/20 pts.** Use the bilinear transform with $T = 2$ to design a Type 2 Chebyshev digital lowpass filter that meets the following specifications:

passband edge freq.	$\omega_p = 0.2\pi$ rad/sample
stopband edge freq.	$\omega_s = 0.8\pi$ rad/sample
max. passband ripple	$\frac{1}{\sqrt{1+\epsilon^2}} = 0.8$
min. stopband atten.	$1/A = 0.2$

Give the digital filter transfer function $H(z)$.

Hint: The design formulas for the analog Type 2 Chebyshev filter are given on pages A-8 and A-9 of the notes file `ECE5213NotesAnalogFilterDesign.pdf` and in Appendix A.3 of the text on page 869.

(a) **4/3 pts.** Use the bilinear transform with $T = 2$ to find the critical analog frequencies Ω_p and Ω_s by prewarping the digital frequencies ω_p and ω_s.

Notes p. 9-7: $\Omega_p = \tan\frac{\omega_p}{2} = \tan\frac{\pi}{10} = 0.324920$ $\boxed{00}$

$\Omega_s = \tan\frac{\omega_s}{2} = \tan\frac{4\pi}{10} = \tan\frac{2\pi}{5} = 3.07768$ $\boxed{01}$

(b) **7/6 pts.** Find the required filter order N.

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{8}{10}$$

$$\sqrt{1+\varepsilon^2} = \frac{10}{8}$$

$$1+\varepsilon^2 = \frac{100}{64}$$

$$\varepsilon^2 = \frac{100}{64} - 1 = \frac{100-64}{64} = \frac{36}{64}$$

$$\varepsilon = \sqrt{\frac{36}{64}} = \frac{6}{8} = \frac{3}{4}$$

$$\frac{1}{A} = \frac{2}{10}$$

$$A = \frac{10}{2} = 5$$

(A.17) $$N = \left\lceil \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil$$

$$= \left\lceil \frac{\cosh^{-1}(\sqrt{25-1}/(3/4))}{\cosh^{-1}(9.47214)} \right\rceil$$

$\boxed{03}$

$$= \left\lceil \frac{\cosh^{-1}(4\sqrt{24}/3)}{2.93870} \right\rceil$$

$\boxed{04}$

$$= \left\lceil \frac{\cosh^{-1}(6.53197)}{2.93870} \right\rceil$$

$$= \left\lceil \frac{2.56394}{2.93870} \right\rceil = \lceil 0.872475 \rceil = 1$$

$$\boxed{N = 1}$$

(c) **7/6 pts.** Find the poles and zeros and give an explicit expression for $H_a(s)$.

(A.22) $z_1 = \frac{j\Omega_s}{\cos\frac{\pi}{2}} = \frac{j\Omega_s}{0} \longrightarrow \infty$ $\Rightarrow$ There are <u>no</u> zeros in the finite S-plane

$\Rightarrow$ See the "Note" on the top of page A-9 of the Notes. This simply means that the denominator will be order 1 in S and the numerator will be order zero (a constant).

(A.24c) $\gamma = A + \sqrt{A^2-1} = 5 + \sqrt{24} = 9.89898$ $\boxed{05}$

$\xi = \frac{\gamma^2+1}{2\gamma} = 5$ $\quad \zeta = \frac{\gamma^2-1}{2\gamma} = 4.89898$ $\boxed{06}$

(A.24b) $\alpha_1 = -\Omega_p \zeta \sin\frac{\pi}{2} = -\Omega_p \zeta = -1.59177$ $\boxed{07}$

$\beta_1 = \Omega_p \xi \cos\frac{\pi}{2} = 0$

(A.24a) $\sigma_1 = \frac{\Omega_s \alpha_1}{\alpha_1^2+\beta_1^2} = \frac{\Omega_s \alpha_1}{\alpha_1^2} = \frac{\Omega_s}{\alpha_1} = -1.93349$ $\boxed{08}$

$\Omega_1 = \frac{-\Omega_s \beta_1}{\alpha_1^2+\beta_1^2} = \frac{-\Omega_s \cdot 0}{\alpha_1^2+0} = 0$

(A.23) $p_1 = \sigma_1 + j\Omega_1 = \sigma_1 = -1.93349$

(A.21) $H_a(s) = \frac{C_0}{s-p_1}$

Notes p. A-9: $H_a(0) = 1 = \frac{C_0}{-p_1} \Rightarrow C_0 = -p_1$

$$\boxed{H_a(s) = \frac{-p_1}{s-p_1} = \frac{1.93349}{s+1.93349}}$$

Problem 5, cont...

(d) **7/5 pts.** Use the bilinear transform to find $H(z)$.

Notes p. 9-7:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{-p_1}{s-p_1}\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{-p_1}{\frac{1-z^{-1}}{1+z^{-1}} - p_1} \cdot \underbrace{\frac{1+z^{-1}}{1+z^{-1}}}_{\text{"one"}}$$

$$= \frac{-(1+z^{-1})p_1}{1-z^{-1}-(1+z^{-1})p_1} = \frac{-p_1 - p_1 z^{-1}}{1-z^{-1}-p_1-p_1 z^{-1}}$$

$$= \frac{-p_1 - p_1 z^{-1}}{(1-p_1)-(1+p_1)z^{-1}} = \frac{\frac{-p_1}{1-p_1} - \frac{p_1}{1-p_1} z^{-1}}{1 - \frac{1+p_1}{1-p_1} z^{-1}}$$

$$= \frac{\frac{1.93349}{1+1.93349} + \frac{1.93349}{1+1.93349} z^{-1}}{1 - \frac{1-1.93349}{1+1.93349} z^{-1}}$$

$$\boxed{H(z) = \frac{0.659109 + 0.659109 z^{-1}}{1 + 0.318219 z^{-1}} \qquad \text{ROC: } |z| > 0.318219}$$