ECE 4213/5213
Test 2

Monday, November 21, 2016
4:30 PM - 5:45 PM

Fall 2016 — Name: SOLUTION

Dr. Havlicek — Student Num:

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work problem 5 and select any three out of problems 1-4. Each problem counts 25 points. Below, circle the numbers of the three problems you wish to have graded out of problems 1-4.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

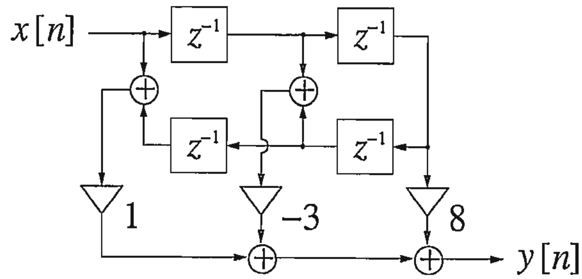
1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ________________ Date: ________________

1. **25/20 pts.** The figure below shows a linear phase FIR structure for a fourth-order Type 1 linear phase FIR filter H.



(a) **9/7 pts.** Find the impulse response $h[n]$.

Direct from diagram: $y[n] = x[n] - 3x[n-1] + 8x[n-2] - 3x[n-3] + x[n-4]$

$$= x[n] * h[n] = \sum_{k=0}^{4} h[k]x[n-k]$$

$$= h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + h[3]x[n-3] + h[4]x[n-4]$$

comparing to the first line: $h[0]=1$, $h[1]=-3$, $h[2]=8$, $h[3]=-3$, $h[4]=1$

$$\boxed{h[n] = \delta[n] - 3\delta[n-1] + 8\delta[n-2] - 3\delta[n-3] + \delta[n-4]}$$

(The answer could have been read directly off the first line)

(b) **8/7 pts.** Find the frequency response $H(e^{j\omega})$.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$\boxed{H(e^{j\omega}) = 1 - 3e^{-j\omega} + 8e^{-j2\omega} - 3e^{-j3\omega} + e^{-j4\omega}}$$

Problem 1, cont...

(c) **8/6 pts.** Find the system group delay $\tau_g(\omega)$.

$$H(e^{j\omega}) = e^{-j2\omega}\left[e^{j2\omega} - 3e^{j\omega} + 8 - 3e^{-j\omega} + e^{-j2\omega}\right]$$

$$= \left\{8 - 6\left[\frac{e^{j\omega}+e^{-j\omega}}{2}\right] + 2\left[\frac{e^{j2\omega}+e^{-j2\omega}}{2}\right]\right\} e^{-j2\omega}$$

$$= \underbrace{\left[8 - 6\cos\omega + 2\cos(2\omega)\right]}_{A(\omega)} \underbrace{e^{-j2\omega}}_{e^{j\theta(\omega)}}$$

$$\theta(\omega) = -2\omega$$

$$\tau_g(\omega) = -\frac{d}{d\omega}\theta(\omega) = -\frac{d}{d\omega}(-2\omega)$$

$$\boxed{\tau_g(\omega) = 2}$$

2. **25/20 pts.** The causal IIR digital filter G has transfer function

$$G(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 2z^{-1}\right)}, \qquad |z| > 2.$$

Note that this filter is undesirable for implementation because $G(z)$ is unstable and does not have minimum phase.

Design a new causal IIR filter H such that (1) H and G have the same magnitude response, *e.g.*, $|H(e^{j\omega})| = |G(e^{j\omega})|\ \forall\, \omega \in \mathbb{R}$, (2) H is causal and stable, and (3) $H(z)$ has minimum phase.

(a) **17/13 pts.** Find the transfer function $H(z)$.

- For stability + causality, all poles must be inside the unit circle.
- For minimum phase, all zeros must be inside the unit circle.
- Here, we have a "bad" zero at $z = -2$ and a "bad" pole at $z = +2$.

→ They must be reflected inside the unit circle.

$$G(z) = \underbrace{\frac{1 - \frac{1}{2}z^{-1}}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}}_{\text{good poles \& zeros}} \cdot \underbrace{\frac{1}{1 - 2z^{-1}}}_{\text{the bad pole}} \cdot \underbrace{\frac{z^{-1} - 2}{z^{-1} - 2}}_{1} \cdot \underbrace{(1 + 2z^{-1})}_{\text{the bad zero}} \cdot \underbrace{\frac{z^{-1} + 2}{z^{-1} + 2}}_{1}$$

$$= \frac{1 - \frac{1}{2}z^{-1}}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})} \cdot \frac{1}{z^{-1} - 2} \cdot \underbrace{\frac{z^{-1} - 2}{1 - 2z^{-1}}}_{\text{all pass}} \cdot (z^{-1} + 2) \cdot \underbrace{\frac{1 + 2z^{-1}}{z^{-1} + 2}}_{\text{all pass}}$$

$$= \underbrace{\frac{(1 - \frac{1}{2}z^{-1})(z^{-1} + 2)}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})(z^{-1} - 2)}}_{H(z)} \cdot \underbrace{\frac{(z^{-1} - 2)(1 + 2z^{-1})}{(1 - 2z^{-1})(z^{-1} + 2)}}_{\text{all pass}}$$

$$H(z) = \frac{\cancel{(1 - \frac{1}{2}z^{-1})}(2)(1 + \frac{1}{2}z^{-1})}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})(-2)\cancel{(1 - \frac{1}{2}z^{-1})}} = -\frac{1 + \frac{1}{2}z^{-1}}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

$$\boxed{H(z) = -\frac{1 + \frac{1}{2}z^{-1}}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}}$$

Problem 2, cont...

(b) **8/7 pts.** Give a direct-form I structure for $H(z)$.

$$H(z) = \frac{-1-\frac{1}{2}z^{-1}}{(1+\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{-1-\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}+\frac{1}{4}z^{-1}-\frac{1}{12}z^{-2}}$$

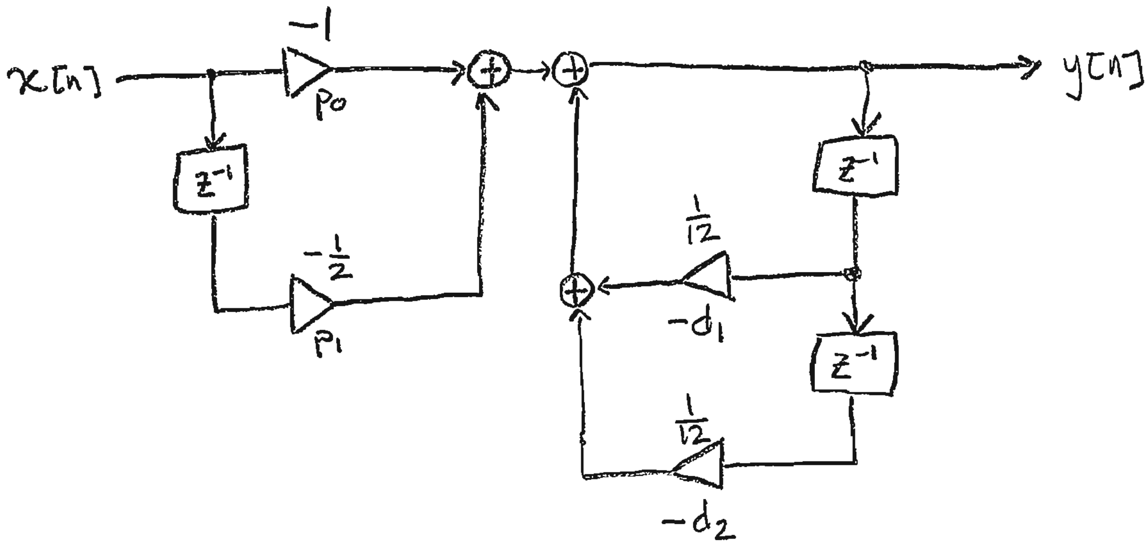
$$= \frac{-1-\frac{1}{2}z^{-1}}{1-\frac{1}{12}z^{-1}-\frac{1}{12}z^{-2}}$$

Book: (8.23), p. 427:

$p_0 = -1 \qquad p_1 = -\frac{1}{2}$

$d_1 = -\frac{1}{12} \quad d_2 = -\frac{1}{12}$

Book: Fig. 8.13(a), p. 429:



3. **25/20 pts.** A causal IIR digital filter H has transfer function

$$H(z) = \frac{1 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{4}z^{-1})}.$$

Give a Parallel Form I realization of $H(z)$.

Hint: Parallel Form I is given on page 432 of the text in equation (8.30) and Fig. 8.19(a). It is also given in the Chapter 8 notes on pages 8-27 and 8-28.

In this problem, you should perform a partial fraction expansion on $H(z)$ to get a sum of two first-order terms. Implement each term using IIR Direct Form II and connect them in parallel as shown in Fig. 8.19(a) on page 432 of the text.

Because $H(z)$ is a proper fraction in this problem, the "direct transmission" term γ_0 shown in Fig. 8.19(a) of the text is zero.

$$H(z) = \frac{1+\frac{1}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{A}{1-\frac{1}{3}z^{-1}} + \frac{B}{1-\frac{1}{4}z^{-1}}$$

$$A = \left.\frac{1+\frac{1}{2}\theta}{1-\frac{1}{4}\theta}\right|_{\theta=3} = \frac{1+\frac{3}{2}}{1-\frac{3}{4}} = \frac{5/2}{1/4} = \frac{20}{2} = 10$$

$$B = \left.\frac{1+\frac{1}{2}\theta}{1-\frac{1}{3}\theta}\right|_{\theta=4} = \frac{1+2}{1-4/3} = \frac{3}{-1/3} = -9$$

$$H(z) = \frac{10}{1-\frac{1}{3}z^{-1}} - \frac{9}{1-\frac{1}{4}z^{-1}}$$

Book: (8.30), p. 432:

$\gamma_0 = 0$

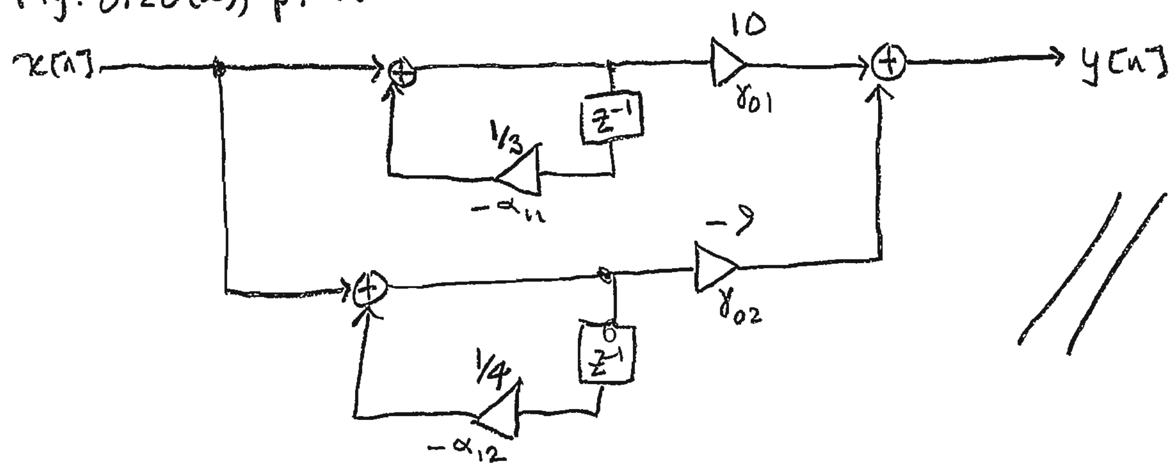
$\gamma_{01} = 10$ $\quad \gamma_{02} = -9$

$\gamma_{11} = 0$ $\quad \gamma_{12} = 0$

$\alpha_{11} = -\frac{1}{3}$ $\quad \alpha_{12} = -1/4$

$\alpha_{21} = 0$ $\quad \alpha_{22} = 0$

Book: Fig. 8.20(a), p. 432:



4. **25/20 pts.** Use the bilinear transform with $T = 2$ to design a digital Type 1 Chebyshev lowpass filter that meets the following specifications:

$$\frac{1}{A} = 0.2 = \frac{2}{10}$$

$$A = \frac{10}{2} = 5$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9 = \frac{9}{10}$$

$$\sqrt{1+\varepsilon^2} = \frac{10}{9}$$

$$1+\varepsilon^2 = \frac{100}{81}$$

$$\varepsilon^2 = \frac{100}{81} - 1 = \frac{100-81}{81} = 19/81$$

$$\varepsilon = \frac{\sqrt{19}}{9} \approx 0.484322$$

Passband Edge Freq.	$\omega_p = \pi/8$ rad/sample
Stopband Edge Freq.	$\omega_s = 2\pi/5$ rad/sample
Max. Passband Ripple	$1/\sqrt{1+\epsilon^2} = 0.9$
Min. Stopband Atten.	$1/A = 0.2$

Hint: The design formulas for the analog Type 1 Chebyshev filter are given on pages A-7 and A-8 of the notes file `ECE5213NotesAnalogFilterDesign.pdf` and in Appendix A.3 of the text on pp. 867-868.

(a) **4/3 pts.** Find the analog edge frequencies Ω_p and Ω_s by using the "ω-form" of the bilinear transform with $T = 2$ to prewarp the digital edge frequencies ω_p and ω_s.

$$\Omega_p = \tan\left(\frac{\omega_p}{2}\right) = \tan\left(\frac{\pi/8}{2}\right) = \tan\left(\frac{\pi}{16}\right) = 0.198912$$

$$\Omega_s = \tan\left(\frac{\omega_s}{2}\right) = \tan\left(\frac{2\pi/5}{2}\right) = \tan\left(\frac{\pi}{5}\right) = 0.726543$$

(b) **5/3 pts.** Find the required filter order N.

(A.17):

$$N = \left\lceil \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\cosh^{-1}(\sqrt{24}/\varepsilon)}{\cosh^{-1}(3.65258)} \right\rceil = \left\lceil \frac{\cosh^{-1}(10.1151)}{\cosh^{-1}(3.65258)} \right\rceil$$

$$= \left\lceil \frac{3.00473}{1.96929} \right\rceil = \lceil 1.52579 \rceil = 2$$

$$\boxed{N = 2}$$

Problem 4, cont...

(c) 8/7 pts. Give an explicit expression for $H_a(s)$.

(A.19b): $\gamma = \left(\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)^{1/N} = \left(\frac{2.11111}{\varepsilon}\right)^{1/2} = \sqrt{4.35890} = 2.08780$

$\zeta = \frac{\gamma^2+1}{2\gamma} = 1.28339$ $\quad \xi = \frac{\gamma^2-1}{2\gamma} = 0.804412$

(A.19a):

$\sigma_1 = -\Omega_p \xi \sin\frac{\pi}{4} = -0.113142$; $\Omega_1 = \Omega_p \zeta \cos\frac{\pi}{4} = 0.180511$

$\sigma_2 = -\Omega_p \xi \sin\frac{3\pi}{4} = -0.113142$; $\Omega_2 = \Omega_p \zeta \cos\frac{3\pi}{4} = -0.180511$

(A.18): $p_1 = \sigma_1 + j\Omega_1 = -0.113142 + j\,0.180511$

$p_2 = \sigma_2 + j\Omega_2 = -0.113142 - j\,0.180511 = p_1^*$

$\mathrm{Re}\{p_1\} = \sigma_1 = -0.113142$

$|p_1|^2 = \sigma_1^2 + \Omega_1^2 = 0.04539$ $\quad$ Notes p. A-8: $C_0 = \frac{1}{\sqrt{1+\varepsilon^2}} = 0.9$

Notes p. A-8:

$$H_a(s) = C_0 \frac{(-p_1)(-p_2)}{(s-p_1)(s-p_2)} = C_0 \frac{(-p_1)(-p_1^*)}{(s-p_1)(s-p_1^*)} = C_0 \frac{p_1 p_1^*}{s^2-(p_1+p_1^*)s+p_1p_1^*}$$

$$= C_0 \frac{|p_1|^2}{s^2-2\mathrm{Re}[p_1]s+|p_1|^2} = 0.9 \frac{0.04539}{s^2-2(-0.113142)s+0.04539}$$

$$\boxed{H_a(s) = \frac{0.04085}{s^2+0.22628s+0.04539}}$$

Problem 4, cont…

(d) **8/7 pts.** Use the "z-form" of the bilinear transform with $T = 2$ to give an explicit expression for $H(z)$.

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{0.9|p_1|^2}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 - 2\mathrm{Re}[p_1]\cdot\frac{1-z^{-1}}{1+z^{-1}} + |p_1|^2} \cdot \frac{(1+z^{-1})^2}{(1+z^{-1})^2}$$

$$= \frac{0.9|p_1|^2(1+z^{-1})^2}{(1-z^{-1})^2 - 2\mathrm{Re}[p_1](1-z^{-1})(1+z^{-1}) + |p_1|^2(1+z^{-1})^2}$$

$$= \frac{0.9|p_1|^2(1+2z^{-1}+z^{-2})}{(1-2z^{-1}+z^{-2}) - 2\mathrm{Re}[p_1](1-z^{-2}) + |p_1|^2(1+2z^{-1}+z^{-2})}$$

$$= \frac{0.9|p_1|^2 + 1.8|p_1|^2 z^{-1} + 0.9|p_1|^2 z^{-2}}{(1-2\mathrm{Re}[p_1]+|p_1|^2) + (-2+2|p_1|^2)z^{-1} + (1+2\mathrm{Re}[p_1]+|p_1|^2)z^{-2}}$$

$$= \frac{0.04085 + 0.08169z^{-1} + 0.04085z^{-2}}{1.27167 - 1.90923z^{-1} + 0.81910z^{-2}}$$

$$= \frac{0.03212 + 0.06424z^{-1} + 0.03212z^{-2}}{1 - 1.50136z^{-1} + 0.64411z^{-2}}$$

$$\boxed{H(z) = \frac{0.03212 + 0.06424z^{-1} + 0.03212z^{-2}}{1 - 1.50136z^{-1} + 0.64411z^{-2}}}$$

5. **25/20 pts.** Use the window design method with an appropriate fixed window from Table 10.2 (p. 540 of the text) to design a causal lowpass FIR digital filter that meets the following specifications:

Passband Edge Freq.	$\omega_p = 0.5\pi$ rad/sample
Stopband Edge Freq.	$\omega_s = 0.7\pi$ rad/sample
Max. Passband Ripple	$\delta_p = 0.010$
Max. Stopband Ripple	$\delta_s = 0.010$

Give the filter impulse response $h[n]$.

$$\alpha_s = -20\log_{10}\delta_s = -20\log_{10} 0.01 = 40 \text{ dB}$$

Table 10.2: Hann, Hamming, and Blackman can meet the stopband spec.

$$\Delta\omega = \omega_s - \omega_p = 0.7\pi - 0.5\pi = 0.2\pi$$

$$\text{Hann: } M = \left\lceil \frac{3.11\pi}{\Delta\omega} \right\rceil = \left\lceil \frac{3.11}{0.2} \right\rceil = \lceil 15.55 \rceil = 16$$

$$\text{Hamming: } M = \left\lceil \frac{3.32\pi}{\Delta\omega} \right\rceil = \left\lceil \frac{3.32}{0.2} \right\rceil = \lceil 16.6 \rceil = 17$$

$$\text{Blackman: } M = \left\lceil \frac{5.56\pi}{\Delta\omega} \right\rceil = \left\lceil \frac{5.56}{0.2} \right\rceil = \lceil 27.8 \rceil = 28$$

Hann meets the spec with the lowest order. $\Rightarrow$ use Hann.

$M = 16$, order $= N = 2M = 32$, length $= 2M+1 = 33$

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.5\pi + 0.7\pi}{2} = \frac{1.2\pi}{2} = 0.6\pi$$

$$(10.14): \quad h_{LP}[n] = \frac{\sin\omega_c n}{\pi n} = \frac{\sin(0.6\pi n)}{\pi n}$$

$$(10.33): \quad w[n] = \frac{1}{2}\left[1 + \cos\left(\frac{\pi}{16}n\right)\right], \quad -16 \le n \le 16$$

$$w[n]h_{LP}[n] = \frac{1}{2}\left[1 + \cos\left(\frac{\pi}{16}n\right)\right]\frac{\sin(0.6\pi n)}{\pi n}, \quad -16 \le n \le 16$$

Shift to make causal:

$$\boxed{h[n] = \frac{1}{2}\left[1 + \cos\frac{\pi(n-16)}{16}\right]\frac{\sin[0.6\pi(n-16)]}{\pi(n-16)}, \quad 0 \le n \le 32}$$