ECE 4213/5213
Test 2

Wednesday, November 29, 2017
4:30 PM - 5:45 PM

Fall 2017 Name: SOLUTION

Dr. Havlicek Student Num:____________

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________

2. (25/20) ________

3. (25/20) ________

4. (25/20) ________

5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:____________ Date:____________

1. **25/20 pts.** A causal discrete-time LTI system G has transfer function

$$G(z) = \frac{(1 - \frac{3}{2}z^{-1})(1 + \frac{1}{3}z^{-1})(1 + \frac{5}{3}z^{-1})}{(1 - z^{-1})^2(1 - \frac{1}{4}z^{-1})}.$$

(a) **5/4 pts.** Is the system G BIBO stable? (Justify your answer)

A causal discrete-time LTI system is BIBO stable iff all the poles are strictly inside the unit circle. $G(z)$ has a pole at $z=1$ that is on the unit circle.

$\Rightarrow$ NOT BIBO STABLE

(b) **5/4 pts.** Does the system G have minimum phase? (Justify your answer)

Minimum phase means that all of the zeros are strictly inside the unit circle.

$G(z)$ has a zero at $z = \frac{3}{2}$ which is outside the unit circle. The zero at $z = -\frac{5}{3}$ is also outside the unit circle.

$\Rightarrow$ NOT MINIMUM PHASE

Problem 1 cont.…

(c) **15/12 pts.** Now consider a second causal discrete-time LTI system H with impulse response

$$h[n] = a^n g[n],$$

where $a \in \mathbb{R}$ is a real constant and where $g[n]$ is the impulse response of the system G from part (a). For what values of the real constant a is the system causal, BIBO stable, and minimum phase?

Since G is causal, we know that $g[n]=0 \quad \forall n<0$.
This means that $h[n]=0 \; \forall n<0$ too, so H is causal.

Then H is stable iff all the poles of $H(z)$ are strictly inside the unit circle.

And H is minimum phase iff all the zeros of $H(z)$ are strictly inside the unit circle.

→ So H is causal, BIBO stable, and minimum phase iff all the poles and all the zeros of $H(z)$ are inside the unit circle.

z-transform table: $a^n g[n] \overset{Z}{\longleftrightarrow} G\left(\frac{z}{a}\right)$.

$$\text{So} \quad H(z) = G\left(\frac{z}{a}\right) = \frac{\left(1-\frac{3a}{2}z^{-1}\right)\left(1+\frac{a}{3}z^{-1}\right)\left(1+\frac{5a}{3}z^{-1}\right)}{\left(1-az^{-1}\right)^2\left(1-\frac{a}{4}z^{-1}\right)}$$

poles: $z = a, \frac{a}{4}$. Both inside the unit circle if $|a|<1$.

zeros: $\frac{3a}{2}, -\frac{a}{3}, -\frac{5a}{3}$. All inside the unit circle provided that $\left|\frac{5a}{3}\right| < 1 \mapsto |a| < \frac{3}{5}$

$\Rightarrow$ All poles and zeros are inside the unit circle provided that

$$\boxed{|a| < \frac{3}{5}}$$

2. **25/20 pts.** H is a causal, stable type III linear phase FIR digital filter with order $N = 6$. The impulse response $h[n]$ is real and has length $N + 1 = 7$.

The value of $H(z)$ at $z = \frac{1}{2}$ is given by $H(\frac{1}{2}) = -39$.

$H(z)$ has a complex zero at $z = \frac{1}{2}e^{j\frac{\pi}{3}}$.

Find the transfer function $H(z)$ and the impulse response $h[n]$.

$N=6 \Rightarrow$ There are 6 zeros.

So $H(z) = C_0 \prod_{m=1}^{6} (1 - \gamma_m z^{-1})$, where C_0 is a constant and γ_m are the zeros.

- Because H is a type III linear phase FIR filter, there must be zeros at $z=1$ $(\omega=0)$ and $z=-1$ $(\omega=\pm\pi)$ (Notes p. 7.49)
- It is given that there is a complex zero @ $z = \frac{1}{2}e^{j\pi/3}$.
- Because H is linear phase, there must be a mirror image zero @ $z = 2e^{-j\pi/3}$.
- Because $h[n]$ is real, all complex zeros must occur in conjugate pairs. So there must be two more zeros @ $z = \frac{1}{2}e^{-j\pi/3}$ and $z = 2e^{j\pi/3}$.
- So the 6 zeros are at $z = 1, -1, \frac{1}{2}e^{j\pi/3}, \frac{1}{2}e^{-j\pi/3}, 2e^{j\pi/3},$ and $2e^{-j\pi/3}$.

$$H(z) = C_0(1-z^{-1})(1+z^{-1})(1-\tfrac{1}{2}e^{j\pi/3}z^{-1})(1-\tfrac{1}{2}e^{-j\pi/3}z^{-1})(1-2e^{j\pi/3}z^{-1}) \cdot (1-2e^{-j\pi/3}z^{-1})$$

$$= C_0(1-z^{-2})\left[1-\tfrac{1}{2}(e^{j\pi/3}+e^{-j\pi/3})z^{-1}+\tfrac{1}{4}z^{-2}\right]\left[1-2(e^{j\pi/3}+e^{-j\pi/3})z^{-1}+4z^{-2}\right]$$

$$= C_0(1-z^{-2})\left[1-\tfrac{1}{2}\cdot 2\cos\tfrac{\pi}{3}z^{-1}+\tfrac{1}{4}z^{-2}\right]\left[1-2\cdot 2\cos\tfrac{\pi}{3}z^{-1}+4z^{-2}\right] \longrightarrow$$

More Workspace for Problem 2... $\cos\frac{\pi}{3} = \frac{1}{2}$, so

$$H(z) = C_0(1-z^{-2})\left[1-\tfrac{1}{2}z^{-1}+\tfrac{1}{4}z^{-2}\right]\left[1-2z^{-1}+4z^{-2}\right]$$

plug in $z=\frac{1}{2}$: $H(\tfrac{1}{2}) = -39 = C_0(1-4)[1-1+1][1-4+16]$

$$= C_0(-3)(1)(13) = -39C_0$$

$$\Rightarrow C_0 = 1$$

$$H(z) = (1-z^{-2})\left[1-\tfrac{1}{2}z^{-1}+\tfrac{1}{4}z^{-2}\right]\left[1-2z^{-1}+4z^{-2}\right]$$

$$= (1-z^{-2})\left\{1-2z^{-1}+4z^{-2}-\tfrac{1}{2}z^{-1}+z^{-2}-2z^{-3}+\tfrac{1}{4}z^{-2}-\tfrac{1}{2}z^{-3}+z^{-4}\right\}$$

$$= (1-z^{-2})\left\{1-\tfrac{5}{2}z^{-1}+\tfrac{21}{4}z^{-2}-\tfrac{5}{2}z^{-3}+z^{-4}\right\}$$

$$= 1-\tfrac{5}{2}z^{-1}+\tfrac{21}{4}z^{-2}-\tfrac{5}{2}z^{-3}+z^{-4}$$
$$\qquad -z^{-2}+\tfrac{5}{2}z^{-3}-\tfrac{21}{4}z^{-4}+\tfrac{5}{2}z^{-5}-z^{-6}$$

$$\underline{\underline{= 1-\tfrac{5}{2}z^{-1}+\tfrac{17}{4}z^{-2}+0z^{-3}-\tfrac{17}{4}z^{-4}+\tfrac{5}{2}z^{-5}-z^{-6}}}, \quad \text{ROC: } |z|>0$$

$$\mathcal{Z}^{-1}: \quad \underline{\underline{h[n] = \delta[n]-\tfrac{5}{2}\delta[n-1]+\tfrac{17}{4}\delta[n-2]-\tfrac{17}{4}\delta[n-4]+\tfrac{5}{2}\delta[n-5]-\delta[n-6]}}$$

Symmetry Check: $\left[1 \ \ -\tfrac{5}{2} \ \ \tfrac{17}{4} \ \ 0 \ \ -\tfrac{17}{4} \ \ \tfrac{5}{2} \ \ -1\right]$ ✓ Type III linear phase FIR

SEE NOTES PAGE 7.58

3. **25/20 pts.** H is a simple digital IIR bandpass filter with order $N = 2$, passband center frequency $\omega_0 = \frac{2\pi}{5}$, and quality $Q = \frac{3}{5}$.

Find the frequency response $H(z)$.

$$Q = \frac{3}{5} = \frac{\omega_0}{B_W} = \frac{2\pi/5}{B_W} \rightarrow 3B_W = 2\pi \rightarrow B_W = \frac{2\pi}{3}$$

$$B_W = \frac{2\pi}{3} = \arccos\left(\frac{2\alpha}{1+\alpha^2}\right) \rightarrow \cos\frac{2\pi}{3} = -\frac{1}{2} = \frac{2\alpha}{1+\alpha^2}$$

$$-\frac{1}{2} - \frac{1}{2}\alpha^2 = 2\alpha \qquad -\frac{1}{2}\alpha^2 - 2\alpha - \frac{1}{2} = 0$$

Quadratic formula: $a = -\frac{1}{2}$, $b = -2$, $c = -\frac{1}{2}$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4-1}}{-1} = -2 \pm \sqrt{3}$$

$\alpha = -0.267949$ or $\alpha = -3.73205$

But, as stated on notes p. 7.58, we must have $|\alpha| < 1$ for stability. So, $\underline{\underline{\alpha = -0.267949}}$

Now, $\omega_0 = \frac{2\pi}{5} = \arccos\beta \rightarrow$ $\underline{\underline{\beta = \cos\frac{2\pi}{5} = 0.309017}}$

Notes p. 7.58:

$$H(z) = \frac{1-\alpha}{2} \, \frac{1 - z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

$$= \frac{1+0.267949}{2} \, \frac{1 - z^{-2}}{1 - 0.309017(1 - 0.267949)z^{-1} - 0.267949\, z^{-2}}$$

⟶

More Workspace for Problem 3...

$$\boxed{H(z) = \frac{0.633975\,(1 - z^{-2})}{1 - 0.226216\,z^{-1} - 0.267949\,z^{-2}}}$$

4. **25/20 pts.** Design an **analog** Butterworth low pass filter to meet the following **analog** specification:

$A = \frac{1}{0.063} = 15.8730$

$1 + \varepsilon^2 = \frac{1}{(0.95)^2}$

$\varepsilon^2 = \frac{1}{(0.95)^2} - 1$

$= 0.108033$

passband edge freq.	$\Omega_p = 2500\pi$ rad/sec
stopband edge freq.	$\Omega_s = 7000\pi$ rad/sec
min. stopband attenuation	$1/A = 0.063$
max. passband attenuation	$1/\sqrt{1+\varepsilon^2} = 0.95$

Give the analog filter transfer function $H_a(s)$.

Hint: The design formulas for the analog Butterworth filter are given on pages A-4 and A-5 of the notes file `ECE5213NotesAnalogFilterDesign.pdf` and in Appendix A.2 on page 865 of the text.

$$\text{(A.9)}: \quad N = \left\lceil \frac{\frac{1}{2}\log_{10}\left[(A^2-1)/\varepsilon^2\right]}{\log_{10} \Omega_s/\Omega_p} \right\rceil = \left\lceil \frac{1.68302}{0.447158} \right\rceil = \lceil 3.763811 \rceil = 4$$

$$\text{(A.8b)}: \quad \frac{1}{A^2} = \frac{1}{1+(\Omega_s/\Omega_c)^8} \Rightarrow A^2 = 1 + \frac{\Omega_s^8}{\Omega_c^8} \Rightarrow A_2 - 1 = \frac{\Omega_s^8}{\Omega_c^8} \Rightarrow \Omega_c^8 = \frac{\Omega_s^8}{A^2-1}$$

$$\Omega_c = \frac{\Omega_s}{(A^2-1)^{1/8}} = \frac{7000\pi}{1.99503} = 3508.72\pi = 11{,}022.98$$

$$\text{(A.11)}: \quad \text{Poles are} \quad p_\ell = \Omega_c e^{j[\pi(N+2\ell-1)/2N]} \quad \text{for} \quad 1 \le \ell \le 4$$

$\ell = 1: \; p_1 = 11{,}022.98 e^{j5\pi/8}$

$\ell = 2: \; p_2 = 11{,}022.98 e^{j7\pi/8}$

$\ell = 3: \; p_3 = 11{,}022.98 e^{j9\pi/8}$

$\ell = 4: \; p_4 = 11{,}022.98 e^{j11\pi/8}$

$$\text{(A.10)}: \quad H_a(s) = \frac{\Omega_c^4}{\prod_{\ell=1}^{4} (s - p_\ell)}$$

$$H_a(s) = \frac{14.763719 \times 10^{15}}{(s - 11{,}022.98e^{j5\pi/8})(s - 11{,}022.98e^{j7\pi/8})(s - 11{,}022.98e^{j9\pi/8})(s - 11{,}022.98e^{j11\pi/8})}$$

More Workspace for Problem 4. . .

5. **25/20 pts.** Use the bilinear transform with $T = 2$ to design a Type 2 Chebyshev digital lowpass filter that meets the following specifications:

$1+\varepsilon^2 = \frac{1}{(0.8)^2} = \frac{1}{0.64} = {}^{100}/_{64}$

$\varepsilon^2 = \frac{100}{64} - 1 = \frac{36}{64}$

$\varepsilon = \frac{6}{8} = 3/4$

passband edge freq.	$\omega_p = 0.2\pi$ rad/sample
stopband edge freq.	$\omega_s = 0.8\pi$ rad/sample
max. passband ripple	$\frac{1}{\sqrt{1+\epsilon^2}} = 0.8$
min. stopband atten.	$1/A = 0.2$

$\frac{1}{A} = \frac{2}{10}$

$2A = 10$

$A = 5$

Give the digital filter transfer function $H(z)$.

Hint: The design formulas for the analog Type 2 Chebyshev filter are given on pages A-8 and A-9 of the notes file `ECE5213NotesAnalogFilterDesign.pdf` and in Appendix A.3 of the text on page 869.

(a) **4/3 pts.** Use the bilinear transform with $T = 2$ to find the critical analog frequencies Ω_p and Ω_s by prewarping the digital frequencies ω_p and ω_s.

Notes p. 9-7:

$$\Omega_p = \tan\frac{\omega_p}{2} = \tan\frac{\pi}{10} = 0.324920$$

$$\Omega_s = \tan\frac{\omega_s}{2} = \tan\frac{4\pi}{10} = \tan\frac{2\pi}{5} = 3.07768$$

(b) **7/6 pts.** Find the required filter order N.

(A.17): $$N = \left\lceil \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\cosh^{-1}(\sqrt{25-1}/(3/4))}{\cosh^{-1}(9.47214)} \right\rceil$$

$$= \left\lceil \frac{\cosh^{-1}(4\sqrt{24}/3)}{2.93870} \right\rceil = \left\lceil \frac{\cosh^{-1}(6.53197)}{2.93870} \right\rceil$$

$$= \left\lceil \frac{2.56394}{2.93870} \right\rceil = \lceil 0.872475 \rceil = \underline{\underline{1}}$$

Problem 5, cont...

(c) **7/6 pts.** Find the poles and zeros and give an explicit expression for $H_a(s)$.

(A.22): $Z_1 = \frac{j\Omega_s}{\cos \pi/2} = \frac{j\Omega_s}{0} \longrightarrow \infty$ There are no zeros in the finite s-plane.

(see "note" on top of Notes p. A-9; this just means that the order of the denominator will be 1 and the order of the numerator will be zero).

(A.24c): $\gamma = A + \sqrt{A^2-1} = 5 + \sqrt{24} = 9.89898$

$\xi = \frac{\gamma^2+1}{2\gamma} = 5$ $\qquad \zeta = \frac{\gamma^2-1}{2\gamma} = 4.89898$

(A.24b): $\alpha_1 = -\Omega_p \zeta \sin\frac{\pi}{2} = -\Omega_p \zeta = -1.59177$; $\beta_1 = \Omega_p \xi \cos\frac{\pi}{2} = 0$

(A.24a): $\sigma_1 = \frac{\Omega_s \alpha_1}{\alpha_1^2+\beta_1^2} = \frac{\Omega_s \alpha_1}{\alpha_1^2} = \frac{\Omega_s}{\alpha_1} = -1.93349$

$\Omega_1 = \frac{-\Omega_s \beta_1}{\alpha_1^2+\beta_1^2} = \frac{-\Omega_s \cdot 0}{\alpha_1^2} = 0$

(A.23): $p_1 = \sigma_1 + j\Omega_1 = \sigma_1 = -1.93349$

(A.21): $H_a(s) = \frac{C_0}{s-p_1}$ $\qquad$ Notes p. A-9: $H_a(0) = 1 = \frac{C_0}{-p_1} \Rightarrow C_0 = -p_1$

$$\boxed{H_a(s) = \frac{-p_1}{s-p_1} = \frac{1.93349}{s+1.93349}}$$

Problem 5, cont…

(d) **7/5 pts**. Use the bilinear transform to find $H(z)$.

Notes p. 9-7: $H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$

$$H(z) = \frac{-p_1}{\frac{1-z^{-1}}{1+z^{-1}} - p_1} \cdot \frac{1+z^{-1}}{1+z^{-1}}$$

$$= \frac{-(1+z^{-1})p_1}{1-z^{-1}-(1+z^{-1})p_1} = \frac{-p_1 - p_1 z^{-1}}{1-z^{-1}-p_1-p_1 z^{-1}}$$

$$= \frac{-p_1 - p_1 z^{-1}}{(1-p_1)-(1+p_1)z^{-1}} = \frac{\frac{-p_1}{1-p_1} - \frac{p_1}{1-p_1} z^{-1}}{1 - \frac{1+p_1}{1-p_1} z^{-1}}$$

$$= \frac{\frac{1.93349}{1+1.93349} + \frac{1.93349}{1+1.93349} z^{-1}}{1 - \frac{1-1.93349}{1+1.93349} z^{-1}}$$

$$\boxed{H(z) = \frac{0.659109 + 0.659109 z^{-1}}{1 + 0.318219 z^{-1}}}$$