ECE 4213/5213
Test 2

Wednesday, November 14, 2018
4:30 PM - 5:45 PM

Fall 2018

Dr. Havlicek

Name: SOLUTION

Student Num:

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ________________ Date: ________________

1. **25/20 pts.** Let $x[n]$ and $h[n]$ be 3-point discrete-time signals given by

$$x[n] = [1 \;\; 2 \;\; 3] = \delta[n] + 2\delta[n-1] + 3\delta[n-2],\; 0 \le n \le 2,$$

$N=3$

and

$$h[n] = [-2 \quad 4 \quad -2] = -2\delta[n] + 4\delta[n-1] - 2\delta[n-2],\; 0 \le n \le 2.$$

Use the DFT to find the 3-point circular convolution $y[n] = x[n] \circledast_3 h[n]$.

$$X[k] = \sum_{n=0}^{2} x[n] W_3^{nk} = 1W_3^{0k} + 2W_3^{k} + 3W_3^{2k} = 1 + 2W_3^k + 3W_3^{2k},\; 0 \le k \le 2$$

$$H[k] = \sum_{n=0}^{2} h[n] W_3^{nk} = -2W_3^{0k} + 4W_3^{k} - 2W_3^{2k} = -2 + 4W_3^k - 2W_3^{2k},\; 0 \le k \le 2$$

$$Y[k] = X[k]H[k] = (1 + 2W_3^k + 3W_3^{2k})(-2 + 4W_3^k - 2W_3^{2k})$$

$$\begin{aligned}
= -2 &+ 4W_3^k - 2W_3^{2k} \\
&- 4W_3^k + 8W_3^{2k} - 4W_3^{3k} \\
&\qquad\quad - 6W_3^{2k} + 12W_3^{3k} - 6W_3^{4k}
\end{aligned}$$

NOTE

$W_3^{3k} = 1$

$W_3^{4k} = W_3^k$

(notes p. 5.71)

$$Y[k] = -2 + 0W_3^k + 0W_3^{2k} + 8W_3^{3k} - 6W_3^{4k}$$

$$= -2 + 0W_3^k + 0W_3^{2k} + 8 - 6W_3^k$$

$$= 6 - 6W_3^k + 0W_3^{2k},\; 0 \le k \le 2 \qquad (*)$$

By definition,

$$Y[k] = \sum_{n=0}^{2} y[n] W_3^{nk} = y[0] + y[1]W_3^k + y[2]W_3^{2k} \qquad (**)$$

Comparing $(*)$ and $(**)$, we obtain

$$\boxed{\begin{aligned} y[n] &= [6 \;\; -6 \;\; 0] \\ &= 6\delta[n] - 6\delta[n-1],\; 0 \le n \le 2 \end{aligned}}$$

2. **25/20 pts.** The causal, stable IIR digital filter G has transfer function

$$G(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - 4z^{-1}\right)}{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{4}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{1}{4}e^{-j\pi/4}z^{-1}\right)}, \qquad |z| > \frac{1}{4}.$$

Note that $G(z)$ does not have minimum phase because there is a zero outside the unit circle at $z = 4$.

Give the transfer function $H(z)$ for a new causal, stable IIR digital filter H such that $H(z)$ has minimum phase and G and H have the same magnitude response, *e.g.*, $|H(e^{j\omega})| = |G(e^{j\omega})|\ \forall\, \omega \in \mathbb{R}$.

As on notes page 7.28, we have

$$G(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{4}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{1}{4}e^{-j\pi/4}z^{-1}\right)} \underbrace{\left(1 - 4z^{-1}\right)}_{\text{bad zero}} \underbrace{\frac{z^{-1} - 4}{z^{-1} - 4}}_{\text{one}}$$

$$= \underbrace{\frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(z^{-1} - 4\right)}{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{4}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{1}{4}e^{-j\pi/4}z^{-1}\right)}}_{H(z)} \cdot \underbrace{\frac{1 - 4z^{-1}}{z^{-1} - 4}}_{\text{all pass}}$$

$$\boxed{H(z) = \frac{-4\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{4}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{1}{4}e^{-j\pi/4}z^{-1}\right)}, \quad |z| > \frac{1}{4}}$$

3. **25/20 pts.** H is a causal, stable type IV linear phase FIR digital filter with order $N = 3$. The impulse response $h[n]$ is real and $h[0] = 1$.

$H(z)$ has a real zero at $z = \frac{2}{3}$.

Find the transfer function $H(z)$ and the impulse response $h[n]$.

Notes p. 7.44: Because there is a zero @ $z = \frac{2}{3}$, there must also be one at $z = \frac{3}{2}$.

Notes p. 7.50: A type IV filter must have a zero at $z = 1$.

$\Rightarrow$ The zeros are $z_1 = \frac{2}{3}$, $z_2 = \frac{3}{2}$, $z_3 = 1$.

$$H(z) = K \prod_{m=1}^{3} (1 - z_m z^{-1}), \text{ where } K \text{ is a constant}$$

$$= K(1 - \tfrac{2}{3} z^{-1})(1 - \tfrac{3}{2} z^{-1})(1 - z^{-1})$$

$$= K(1 - \tfrac{13}{6} z^{-1} + z^{-2})(1 - z^{-1})$$

$$= K(1 - \tfrac{13}{6} z^{-1} + z^{-2} - z^{-1} + \tfrac{13}{6} z^{-2} - z^{-3})$$

$$= K(1 - \tfrac{19}{6} z^{-1} + \tfrac{19}{6} z^{-2} - z^{-3})$$

$$= K - \tfrac{19}{6} K z^{-1} + \tfrac{19}{6} K z^{-2} - K z^{-3}, \quad \text{ROC: } |z| > 0.$$

Table: $h[n] = K\delta[n] - \frac{19}{6} K \delta[n-1] + \frac{19}{6} K \delta[n-2] - K\delta[n-3]$

$$\Rightarrow h[0] = 1 = K$$

$$\boxed{\begin{aligned} H(z) &= 1 - \tfrac{19}{6} z^{-1} + \tfrac{19}{6} z^{-2} - z^{-3}, \quad |z| > 0 \\ h[n] &= \delta[n] - \tfrac{19}{6}\delta[n-1] + \tfrac{19}{6}\delta[n-2] - \delta[n-3] \end{aligned}}$$

Check: $h[n] = [1 \;\; -\frac{19}{6} \;\; \frac{19}{6} \;\; -1]$

Type IV ✓

$\longrightarrow$ Numbers in boxes like $\boxed{00}$ are calculator registers.

4. **25/20 pts.** Design an **analog** Butterworth low pass filter to meet the following **analog** specification:

$\frac{1}{A} = 0.01$

$A = 100$

passband edge freq.	$\Omega_p = 1000\pi$ rad/sec
stopband edge freq.	$\Omega_s = 6500\pi$ rad/sec
min. stopband attenuation	$1/A = 0.01$
max. passband attenuation	$1/\sqrt{1+\varepsilon^2} = 0.90$

$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9$

$\sqrt{1+\varepsilon^2} = \frac{1}{0.9}$

$1+\varepsilon^2 = \left(\frac{1}{0.9}\right)^2$

$\varepsilon^2 = \left(\frac{1}{0.9}\right)^2 - 1$

$\varepsilon = \sqrt{\left(\frac{1}{0.9}\right)^2 - 1} = \overset{\boxed{50}}{0.484322}$

$\Omega_p \; \boxed{54}$

$\Omega_s \; \boxed{53}$

Give the analog filter transfer function $H_a(s)$.

Hint: The design formulas for the analog Butterworth filter are given on pages A-4 and A-5 of the notes file `ECE5213NotesAnalogFilterDesign.pdf` and in Appendix A.2 on pages 865-866 of the text.

Hint: Make sure that your calculator is set for **radians** and **not** degrees!

$$(A.9): \left\lceil \frac{\frac{1}{2}\log_{10}\frac{A^2-1}{\varepsilon^2}}{\log_{10}\Omega_s/\Omega_p} \right\rceil = \left\lceil \frac{\frac{1}{2}\log_{10} \overset{\boxed{51}}{42{,}627.32}}{\log_{10}\frac{6500\pi}{1000\pi}} \right\rceil = \left\lceil \frac{\overset{\boxed{52}}{2.31484}}{\underset{\boxed{55}}{0.812913}} \right\rceil = \lceil \overset{\boxed{56}}{2.84759} \rceil$$

$$\longrightarrow \underline{N=3}$$

$$(A.8b): \frac{1}{A^2} = \frac{1}{1+(\Omega_s/\Omega_c)^6} \Rightarrow A^2 = 1 + (\Omega_s/\Omega_c)^6 \Rightarrow A^2 - 1 = \frac{\Omega_s^6}{\Omega_c^6}$$

$$\Rightarrow \Omega_c^6 = \frac{\Omega_s^6}{A^2-1} \Rightarrow \Omega_c = \left[\frac{\Omega_s^6}{A^2-1}\right]^{1/6} = \frac{\Omega_s}{(A^2-1)^{1/6}}$$

$$= \frac{\overset{\boxed{53}}{6500\pi}}{\underset{\boxed{57}}{4.64151}} = \overset{\boxed{58}}{4399.505}$$

$$(A.11): p_1 = \Omega_c e^{j\pi(3+2-1)/6} = \Omega_c e^{j4\pi/6} = \Omega_c e^{j2\pi/3}$$

$$p_2 = \Omega_c e^{j\pi(3+4-1)/6} = \Omega_c e^{j6\pi/6} = \Omega_c e^{j\pi} = -\Omega_c$$

$$p_3 = \Omega_c e^{j\pi(3+6-1)/6} = \Omega_c e^{j8\pi/6} = \Omega_c e^{j4\pi/3}$$

$$= \Omega_c e^{j\left(\frac{4\pi}{3}-2\pi\right)} = \Omega_c e^{j\left(\frac{4\pi}{3}-\frac{6\pi}{3}\right)}$$

$$= \Omega_c e^{-j2\pi/3} = p_1^*$$

More Workspace for Problem 4...

$$(A.10):\ H_a(s) = \frac{\Omega_c^N}{\prod_{\ell=1}^{3}(s-p_\ell)} = \frac{\Omega_c^3}{(s-p_1)(s-p_2)(s-p_3)}$$

$$= \frac{\Omega_c^3}{(s-p_1)(s-p_1^*)(s-p_2)} = \frac{\Omega_c^3}{[s^2-(p_1+p_1^*)s+|p_1|^2](s-p_2)}$$

$$= \frac{\Omega_c^3}{(s^2-2\mathrm{Re}\{p_1\}s+|p_1|^2)(s-p_2)} = \frac{\Omega_c^3}{(s^2-2\Omega_c\cos\frac{2\pi}{3}s+\Omega_c^2)(s-p_2)}$$

$$= \frac{\Omega_c^3}{s^3-2\Omega_c\cos\frac{2\pi}{3}s^2+\Omega_c^2 s-p_2 s^2+2\Omega_c p_2\cos\frac{2\pi}{3}s-\Omega_c^2 p_2}$$

$$= \frac{\Omega_c^3}{s^3-(2\Omega_c\cos\frac{2\pi}{3}+p_2)s^2+(\Omega_c^2+2\Omega_c p_2\cos\frac{2\pi}{3})s-\Omega_c^2 p_2} \qquad (p_2=-\Omega_c)$$

$$= \frac{\Omega_c^3}{s^3-(2\Omega_c\cos\frac{2\pi}{3}-\Omega_c)s^2+(\Omega_c^2-2\Omega_c^2\cos\frac{2\pi}{3})s+\Omega_c^3} \qquad (\cos\frac{2\pi}{3}=-\frac{1}{2})$$

$$= \frac{\Omega_c^3}{s^3+2\Omega_c s^2+2\Omega_c^2 s+\Omega_c^3} \qquad (\Omega_c = \overset{\boxed{5.8}}{4{,}399.505})$$

$$\boxed{H_a(s) = \frac{85.15525\times10^9}{s^3+8{,}799.01s^2+38.71129\times10^6 s+85.15525\times10^9}}$$

→ Numbers in boxes like $\boxed{00}$ are calculator registers

5. **25/20 pts.** Use the bilinear transform with $T = 2$ to design a Type 1 Chebyshev digital lowpass filter that meets the following specifications:

$\frac{1}{A} = 0.0316$

$A = 31.64557$ $\boxed{11}$

passband edge freq.	$\omega_p = 0.200\pi$ rad/sample
stopband edge freq.	$\omega_s = 0.766\pi$ rad/sample
max. passband ripple	$\frac{1}{\sqrt{1+\epsilon^2}} = 0.9500$
min. stopband atten.	$1/A = 0.0316$

$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.95$

$\sqrt{1+\varepsilon^2} = \frac{1}{0.95}$

$1+\varepsilon^2 = \left(\frac{1}{0.95}\right)^2$

$\varepsilon^2 = \left(\frac{1}{0.95}\right)^2 - 1$

$\varepsilon = \sqrt{\left(\frac{1}{0.95}\right)^2 - 1}$

$\varepsilon = 0.3286841$ $\boxed{09}$

Give the digital filter transfer function $H(z)$.

Hint: The design formulas for the analog Type 1 Chebyshev filter are given on pages A-7 and A-8 of the notes file ECE5213NotesAnalogFilterDesign.pdf and in Appendix A.3 of the text on pages 867-868.

Hint: Make sure that your calculator is set for **radians** and **not** degrees!

Notes p. 9-7: $\Omega_p = \tan\left(\frac{\omega_p}{2}\right) = \overset{\boxed{04}}{0.3249197}$

$\Omega_s = \tan\left(\frac{\omega_s}{2}\right) = \overset{\boxed{05}}{2.596957}$

(A.17): $N = \left\lceil \frac{\cosh^{-1}\left(\sqrt{A^2-1}/\varepsilon\right)}{\cosh^{-1}\left(\Omega_s/\Omega_p\right)} \right\rceil = \left\lceil \frac{\cosh^{-1}(\overset{\boxed{13}}{96.2315})}{\cosh^{-1}(\underset{\boxed{06}}{7.992613})} \right\rceil = \left\lceil \frac{\overset{\boxed{14}}{5.259877}}{\underset{\boxed{07}}{2.767728}} \right\rceil$

$= \lceil 1.900431 \rceil = 2 \implies \boxed{N=2}$

(A.19b): $\gamma = \left(\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)^{1/2} = \sqrt{6.244998} = \overset{\boxed{15}}{2.5}$

$\zeta = \frac{\gamma^2+1}{2\gamma} = \overset{\boxed{16}}{1.45}$ $\xi = \frac{\gamma^2-1}{2\gamma} = \overset{\boxed{17}}{1.05}$

(A.19a):

$\sigma_1 = -\Omega_p \xi \sin\frac{\pi}{4} = \overset{\boxed{19}}{-0.24124}$

$\sigma_2 = -\Omega_p \xi \sin\frac{3\pi}{4} = -0.24124 = \sigma_1$

$\Omega_1 = \Omega_p \zeta \cos\frac{\pi}{4} = \overset{\boxed{20}}{0.33314}$

$\Omega_2 = \Omega_p \zeta \cos\frac{3\pi}{4} = -0.33314 = -\Omega_1$

(A.18):

$p_1 = \sigma_1 + j\Omega_1 = -0.24124 + j0.33314$

$p_2 = \sigma_2 + j\Omega_2 = \sigma_1 - j\Omega_1 = p_1^*$

Notes p. A-8:

$C_0 = \frac{1}{\sqrt{1+\varepsilon^2}} = 0.95$

More Workspace for Problem 5...

Notes p. A-8: $$H_a(s) = C_0 \frac{(-p_1)(-p_2)}{(s-p_1)(s-p_2)} = C_0 \frac{p_1 p_1^*}{(s-p_1)(s-p_1^*)}$$

$$H_a(s) = C_0 \frac{p_1 p_1^*}{s^2 - (p_1+p_1^*)s + p_1 p_1^*}$$

$$= \frac{C_0 |p_1|^2}{s^2 - 2\mathrm{Re}\{p_1\} s + |p_1|^2} = \frac{C_0(\sigma_1^2+\Omega_1^2)}{s^2 - 2\sigma_1 s + (\sigma_1^2+\Omega_1^2)}$$

$$\boxed{H_a(s) = \frac{\overset{\boxed{23}}{0.16072}}{s^2 + \underset{\boxed{21}}{0.48248}s + \underset{\boxed{22}}{0.16918}}}$$

Notes p. 9-7:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{C_0(\sigma_1^2+\Omega_1^2)}{s^2 - 2\sigma_1 s + (\sigma_1^2+\Omega_1^2)}\Bigg|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{C_0(\sigma_1^2+\Omega_1^2)}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 - 2\sigma_1\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + (\sigma_1^2+\Omega_1^2)} \cdot \underbrace{\frac{(1+z^{-1})^2}{(1+z^{-1})^2}}_{\text{one}}$$

$$= \frac{C_0(\sigma_1^2+\Omega_1^2)(1+z^{-1})^2}{(1-z^{-1})^2 - 2\sigma_1(1-z^{-1})(1+z^{-1}) + (\sigma_1^2+\Omega_1^2)(1+z^{-1})^2}$$

→

More Workspace for Problem 5…

$$\ldots\; H(z) = \frac{c_0(\sigma_1^2+\Omega_1^2)(1+2z^{-1}+z^{-2})}{1-2z^{-1}+z^{-2}-2\sigma_1(1-z^{-2})+(\sigma_1^2+\Omega_1^2)(1+2z^{-1}+z^{-2})}$$

$$= \frac{c_0(\sigma_1^2+\Omega_1^2)(1+2z^{-1}+z^{-2})}{1-2z^{-1}+z^{-2}-2\sigma_1+2\sigma_1 z^{-2}+(\sigma_1^2+\Omega_1^2)+2(\sigma_1^2+\Omega_1^2)z^{-1}+(\sigma_1^2+\Omega_1^2)z^{-2}}$$

$$= \frac{\overbrace{c_0(\sigma_1^2+\Omega_1^2)}^{\boxed{23}}(1+2z^{-1}+z^{-2})}{(1-2\underbrace{\sigma_1}_{\boxed{21}}+\underbrace{\sigma_1^2+\Omega_1^2}_{\boxed{22}})+(-2+2\underbrace{(\sigma_1^2+\Omega_1^2)}_{\boxed{22}})z^{-1}+(1+\underbrace{2\sigma_1}_{-\boxed{21}}+\underbrace{\sigma_1^2+\Omega_1^2}_{\boxed{22}})z^{-2}}$$

plugging in the numbers:

$$H(z) = \frac{\overbrace{0.16072}^{\boxed{23}}+\overbrace{0.32144}^{\boxed{26}}z^{-1}+\overbrace{0.16072}^{\boxed{23}}z^{-2}}{\underbrace{1.65166}_{\boxed{24}}-\underbrace{1.66164}_{\boxed{25}}z^{-1}+\underbrace{0.68670}_{\boxed{27}}z^{-2}}$$

$$\boxed{H(z) = \frac{0.09731+0.19462z^{-1}+0.09731z^{-2}}{1-1.00604z^{-1}+0.41576z^{-2}}}$$