ECE 4213/5213
Test 2

Monday, November 25, 2019
4:30 PM - 5:45 PM

Fall 2019

Dr. Havlicek

Name: SOLUTION

Student Num:

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________

2. (25/20) ________

3. (25/20) ________

4. (25/20) ________

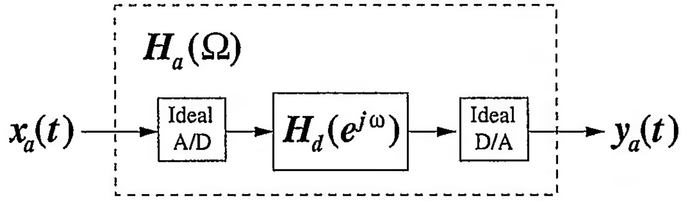
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ________________ Date: ________________

1. **25/20 pts.** The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter.



The sampling frequency of the A/D and D/A converters is $F_T = 5$ kHz, so $\Omega_T = 10,000\pi$ rad/sec. All input signals $x_a(t)$ are bandlimited to $|\Omega| < \Omega_T/2$ (this simply ensures that the overall structure shown in the figure will be an LTI system).

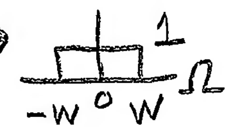
The impulse response of the continuous-time filter H_a is given by

$$h_a(t) = \delta(t) - \frac{\sin 2000\pi t}{\pi t}.$$

Table:

$\delta(t) \longleftrightarrow 1$

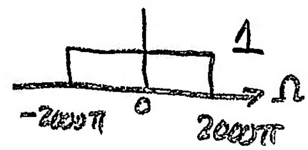
$\frac{\sin Wt}{\pi t} \longleftrightarrow$



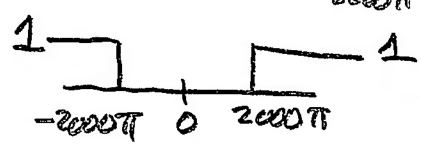
$W = 2000\pi$

(a) **8/6 pts.** Find the continuous-time frequency response $H_a(\Omega)$.

Table: $H_a(\Omega) = 1 -$



$=$



$$\boxed{H_a(\Omega) = \begin{cases} 1, & |\Omega| \geq 2000\pi \\ 0, & |\Omega| < 2000\pi \end{cases}}$$

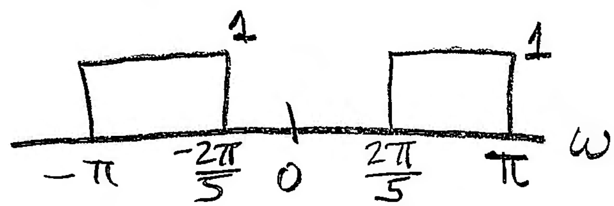
(b) **8/7 pts.** Find the discrete-time frequency response $H_d(e^{j\omega})$.

$T = \frac{1}{5000}$

$\omega = \Omega T = \frac{\Omega}{5000}$

$H_d(e^{j\omega}) = H_a\left(\frac{\omega}{T}\right), \quad |\omega| < \pi$

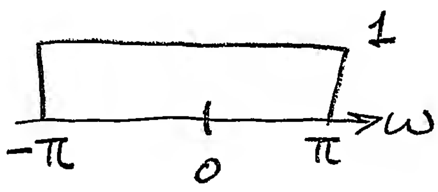
$$\longrightarrow \boxed{H_d(e^{j\omega}) = \begin{cases} 1, & \frac{2\pi}{5} < |\omega| < \pi \\ 0, & |\omega| < \frac{2\pi}{5} \end{cases}}$$



Problem 1, cont...

(c) **9/7 pts**. Find the discrete-time impulse response $h_d[n]$.

$H_d(e^{j\omega}) =$



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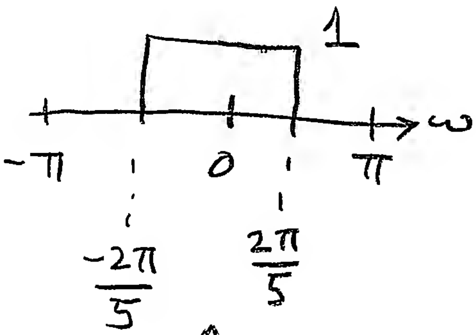

Table

$\delta[n]$

Table

$$\frac{\sin \frac{2\pi}{5} n}{\pi n}$$

$$\boxed{h_d[n] = \delta[n] - \frac{\sin \frac{2\pi}{5} n}{\pi n}}$$

2. **25/20 pts.** Let $x[n]$ and $h[n]$ be finite-length discrete-time signals given by

$$\begin{aligned} x[n] &= [8\ \ 1\ \ 4\ \ 7] \\ &= 8\delta[n] + \delta[n-1] + 4\delta[n-2] + 7\delta[n-3],\ 0 \le n \le 3, \end{aligned} \qquad N_1 = 4$$

and

$$\begin{aligned} h[n] &= [1\ \ -3\ \ \ 2] \\ &= \delta[n] - 3\delta[n-1] + 2\delta[n-2],\ 0 \le n \le 2. \end{aligned} \qquad N_2 = 3$$

Use the DFT to find the circular convolution $y[n] = x[n] \circledast_4 h[n]$. $\Rightarrow$ zero pad $h[n]$ to length 4.

$\Rightarrow$ Will be using W_4

$$X[k] = \sum_{n=0}^{3} x[n] W_4^{nk}$$

$$= 8 + W_4^{k} + 4W_4^{2k} + 7W_4^{3k}$$

$$H[k] = \sum_{n=0}^{3} h[n] W_4^{nk} = 1 - 3W_4^{k} + 2W_4^{2k}$$

$$Y[k] = H[k]\,X[k] = (1 - 3W_4^{k} + 2W_4^{2k})(8 + W_4^{k} + 4W_4^{2k} + 7W_4^{3k})$$

$$\begin{aligned} = 8 \ + W_4^{k} &+ 4W_4^{2k} + 7W_4^{3k} \\ - 24W_4^{k} &- 3W_4^{2k} - 12W_4^{3k} - 21W_4^{4k} \\ &+ 16W_4^{2k} + 2W_4^{3k} + 8W_4^{4k} + 14W_4^{5k} \end{aligned}$$

$$Y[k] = 8 - 23W_4^{k} + 17W_4^{2k} - 3W_4^{3k} - 13W_4^{4k} + 14W_4^{5k}$$

But: $W_4^{4k} = 1 \longmapsto -13W_4^{4k} = -13$

$W_4^{5k} = W_4^{k} \longmapsto 14W_4^{5k} = 14W_4^{k}$

$\Longrightarrow$

More Workspace for Problem 2...

So $Y[k] = 8 - 23W_4^k + 17W_4^{2k} - 3W_4^{3k}$
$- 13 + 14W_4^k$

$$= -5 - 9W_4^k + 17W_4^{2k} - 3W_4^{3k} \qquad (*)$$

- By definition, $Y[k] = \sum_{n=0}^{3} y[n] W_4^{nk}$

$$= y[0] + y[1]W_4^k + y[2]W_4^{2k} + y[3]W_4^{3k} \qquad (**)$$

- Comparing $(*)$ and $(**)$, we obtain:

$$\boxed{\begin{array}{c} y[n] = [-5 \quad -9 \quad 17 \quad -3] \\ \\ y[n] = -5\delta[n] - 9\delta[n-1] + 17\delta[n-2] - 3\delta[n-3], \\ 0 \leq n \leq 3 \end{array}}$$

3. **25/20 pts.** The causal IIR digital filter G has transfer function

$$G(z) = \frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{2}z^{-1}\right)}{\left(1+\frac{1}{2}e^{j\pi/5}z^{-1}\right)\left(1+\frac{1}{2}e^{-j\pi/5}z^{-1}\right)\left(1-\frac{3}{2}z^{-1}\right)}, \qquad |z| > \frac{3}{2}.$$

Find the transfer function $H(z)$ for a *new* digital filter H such that:

(a) $H(z)$ has the same magnitude response as $G(z)$; i.e., $|H(e^{j\omega})| = |G(e^{j\omega})|\ \forall\, \omega \in \mathbb{R}$.

(b) $H(z)$ is causal, stable, and minimum phase.

- For $H(z)$ to be causal and stable, the poles must all be inside the unit circle.
- For $H(z)$ to be minimum phase, the zeros must all be inside the unit circle.

$\Rightarrow$ The zero @ $z=-\frac{3}{2}$ and the pole @ $z=+\frac{3}{2}$ must be reflected inside the unit circle.

$\rightarrow$ Proceeding as on Notes p. 7.28, we get

$$G(z) = \frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}{\left(1+\frac{1}{2}e^{j\pi/5}z^{-1}\right)\left(1+\frac{1}{2}e^{-j\pi/5}z^{-1}\right)} \cdot \underbrace{\left(1+\frac{3}{2}z^{-1}\right)}_{\text{Bad zero}} \cdot \underbrace{\frac{z^{-1}+\frac{3}{2}}{z^{-1}+\frac{3}{2}}}_{\text{one}} \cdot \underbrace{\frac{1}{1-\frac{3}{2}z^{-1}}}_{\text{Bad Pole}} \cdot \underbrace{\frac{z^{-1}-\frac{3}{2}}{z^{-1}-\frac{3}{2}}}_{\text{one}}$$

$$= \underbrace{\frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(z^{-1}+\frac{3}{2}\right)}{\left(1+\frac{1}{2}e^{j\pi/5}z^{-1}\right)\left(1+\frac{1}{2}e^{-j\pi/5}z^{-1}\right)\left(z^{-1}-\frac{3}{2}\right)}}_{H(z)} \cdot \underbrace{\frac{\left(1+\frac{3}{2}z^{-1}\right)\left(z^{-1}-\frac{3}{2}\right)}{\left(z^{-1}+\frac{3}{2}\right)\left(1-\frac{3}{2}z^{-1}\right)}}_{\text{ALL PASS}}$$

$\longrightarrow$

More Workspace for Problem 3...

$$H(z) = \frac{(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})(z^{-1}+3/2)}{(1+\frac{1}{2}e^{j\pi/5}z^{-1})(1+\frac{1}{2}e^{-j\pi/5}z^{-1})(z^{-1}-\frac{3}{2})}$$

$$= \frac{(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})(3/2)(1+\frac{2}{3}z^{-1})}{(1+\frac{1}{2}e^{j\pi/5}z^{-1})(1+\frac{1}{2}e^{-j\pi/5}z^{-1})(-\frac{3}{2})(1-\frac{2}{3}z^{-1})}$$

$$\boxed{H(z) = \frac{-(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})(1+\frac{2}{3}z^{-1})}{(1+\frac{1}{2}e^{j\pi/5}z^{-1})(1+\frac{1}{2}e^{-j\pi/5}z^{-1})(1-\frac{2}{3}z^{-1})} \quad \text{ROC: } |z| > \frac{2}{3}}$$

4. **25/20 pts.** H is a causal, stable type I linear phase FIR digital filter with order $N = 4$. The impulse response $h[n]$ is real and $h[4] = 4$.

$H(z)$ has a zero at $z = 2e^{j\frac{2\pi}{3}}$.

Find the impulse response $h[n]$.

Notes p. 7.45: Because the zero @ $z_1 = 2e^{j2\pi/3}$ is complex, there must also be zeros at the conjugate and mirror locations... i.e., there must also be zeros @:

$$\left.\begin{aligned} z_2 &= z_1^* = 2e^{-j2\pi/3} \\ z_3 &= 1/z_1 = \tfrac{1}{2}e^{-j2\pi/3} \\ z_4 &= z_3^* = \tfrac{1}{2}e^{j2\pi/3} \end{aligned}\right\}$$

Since $N=4$, there are 4 zeros total... in other words, this is all of them.

$$H(z) = K(1-z_1z^{-1})(1-z_2z^{-1})(1-z_3z^{-1})(1-z_4z^{-1})$$

$$= K(1-2e^{j2\pi/3}z^{-1})(1-2e^{-j2\pi/3}z^{-1})(1-\tfrac{1}{2}e^{j2\pi/3}z^{-1})(1-\tfrac{1}{2}e^{-j2\pi/3}z^{-1})$$

$$= K(1-[4\underbrace{\cos\overbrace{\tfrac{2\pi}{3}}^{-1/2}}_{-2}]z^{-1}+4z^{-2})(1-[\underbrace{\cos\tfrac{2\pi}{3}}_{-1/2}]z^{-1}+\tfrac{1}{4}z^{-2})$$

$$= K(1+2z^{-1}+4z^{-2})(1+\tfrac{1}{2}z^{-1}+\tfrac{1}{4}z^{-2})$$

$$\begin{aligned} = K(1 &+ \tfrac{1}{2}z^{-1} + \tfrac{1}{4}z^{-2} \\ &+ 2z^{-1} + z^{-2} + \tfrac{1}{2}z^{-3} \\ &+ 4z^{-2} + 2z^{-3} + z^{-4}) \end{aligned}$$

$$= K(1+\tfrac{5}{2}z^{-1}+\tfrac{21}{4}z^{-2}+\tfrac{5}{2}z^{-3}+z^{-4}), \quad |z|>0$$

→

More Workspace for Problem 4...

Table:

$$h[n] = K\delta[n] + \frac{5K}{2}\delta[n-1] + \frac{21K}{4}\delta[n-2] + \frac{5K}{2}\delta[n-3] + K\delta[n-4]$$

$$h[4] = K$$

- Since it was given that $h[4]=4$, we get $\underline{\underline{K=4}}$

$$\boxed{h[n] = 4\delta[n] + 10\delta[n-1] + 21\delta[n-2] + 10\delta[n-3] + 4\delta[n-4]}$$

5. **25/20 pts.** Design an analog Type 1 Chebyshev low pass filter to meet the following analog design specification:

Numbers in boxes are calculator registers

$\frac{1}{A} = 0.063$ $\boxed{02}$

$A = \frac{1}{0.063} = 15.8730$

passband edge freq.	$\Omega_p = 1000\pi$ rad/sec
stopband edge freq.	$\Omega_s = 3000\pi$ rad/sec
min. stopband attenuation	$1/A = 0.063$
max. passband attenuation	$1/\sqrt{1+\varepsilon^2} = 0.95$

$\sqrt{1+\varepsilon^2} = \frac{1}{0.95}$

$1+\varepsilon^2 = \left(\frac{100}{95}\right)^2$

$\varepsilon = \left[\left(\frac{100}{95}\right)^2 - 1\right]^{1/2}$

Give the analog filter transfer function $H_a(s)$.

$\varepsilon = 0.32868$ $\boxed{03}$

Hint: The design formulas for the analog Type 1 Chebyshev filter are given on pages A-7 and A-8 of the notes file `ECE5213NotesAnalogFilterDesign.pdf` and in Appendix A.3 of the text on pages 867 and 868.

Hint: Make sure that your calculator is set for **radians** and **not** degrees!

$$(A.17):\ N = \left\lceil \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\cosh^{-1}(\overset{\boxed{04}}{48.1967})}{\cosh^{-1}(3)} \right\rceil = \left\lceil \frac{\overset{\boxed{05}}{4.56833}}{\underset{\boxed{06}}{1.76275}} \right\rceil$$

$$= \lceil 2.59160 \rceil = 3 \,//\,.$$

$$(A.19b):\ \gamma = \left(\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)^{1/3} = 1.84152 \ \boxed{07}$$

$$\zeta = \frac{\gamma^2+1}{2\gamma} = 1.19228 \ \boxed{08} \qquad \xi = \frac{\gamma^2-1}{2\gamma} = 0.649248 \ \boxed{09}$$

$$(A.19a):\ \sigma_1 = -\Omega_p \xi \sin\frac{\pi}{6} = -1000\pi\,\xi \sin\frac{\pi}{6} = -1019.84 \ \boxed{10}$$

$$\sigma_2 = -\Omega_p \xi \sin\frac{3\pi}{6} = -1000\pi\,\xi \sin\frac{\pi}{2} = -2039.67 \ \boxed{11}$$

$$\sigma_3 = -\Omega_p \xi \sin\frac{5\pi}{6} = -1000\pi\,\xi \sin\frac{5\pi}{6} = -1019.84 \ \boxed{10}$$

$$\Omega_1 = \Omega_p \zeta \cos\frac{\pi}{6} = 1000\pi\,\zeta \cos\frac{\pi}{6} = 3243.83 \ \boxed{12}$$

$$\Omega_2 = \Omega_p \zeta \cos\frac{3\pi}{6} = 1000\pi\,\zeta \cos\frac{\pi}{2} = 0$$

$$\Omega_3 = \Omega_p \zeta \cos\frac{5\pi}{6} = 1000\pi\,\zeta \cos\frac{5\pi}{6} = -3243.83 \ \boxed{13}$$

$\longrightarrow$

(A.18):

More Workspace for Problem 5...

$$\begin{cases} p_1 = \sigma_1 + j\Omega_1 = -1019.84 + j3243.83 \\ p_2 = \sigma_2 + j\Omega_2 = -2039.67 \\ p_3 = \sigma_3 + j\Omega_3 = \sigma_1 - j\Omega_1 = p_1^* \end{cases}$$

Notes p. A-8:

N odd $\rightarrow C_0 = 1$

$$H_a(\Omega) = C_0 \prod_{\ell=1}^{3} \frac{-p_\ell}{s - p_\ell} = \frac{(-p_1)(-p_2)(-p_3)}{(s-p_1)(s-p_2)(s-p_3)}$$

$$= \frac{-p_1 p_1^* p_2}{(s-p_1)(s-p_1^*)(s-p_2)} = \frac{-|p_1|^2 p_2}{(s^2 - 2\mathrm{Re}[p_1]s + |p_1|^2)(s-p_2)}$$

$$= \frac{-(\sigma_1^2 + \Omega_1^2)\sigma_2}{[s^2 - 2\sigma_1 s + (\sigma_1^2 + \Omega_1^2)](s - \sigma_2)}$$

$$= \frac{-(\sigma_1^2 + \Omega_1^2)\sigma_2}{s^3 - \sigma_2 s^2 - 2\sigma_1 s^2 + 2\sigma_1\sigma_2 s + (\sigma_1^2 + \Omega_1^2)s - (\sigma_1^2 + \Omega_1^2)\sigma_2}$$

$\boxed{14}$

$$= \frac{-(\sigma_1^2 + \Omega_1^2)\sigma_2}{s^3 - (2\sigma_1 + \sigma_2)s^2 + [2\sigma_1\sigma_2 + (\sigma_1^2 + \Omega_1^2)]s - (\sigma_1^2 + \Omega_1^2)\sigma_2}$$

$$\boxed{H_a(s) = \frac{23.5836 \times 10^9}{s^3 + 4079.34 s^2 + 15{,}722{,}730.43 s + 23.5836 \times 10^9}}$$