# ECE 4213/5213
# Test 2

Wednesday, November 18 – Friday, November 20, 2020

Fall 2020 Name: SOLUTION

Dr. Havlicek Student Num:

**Directions:** This test is open book and open notes. You may also use your calculator and the course formula sheet. Other materials are not allowed. You have 60 hours to complete the test. All work must be your own. You may work the test on this test paper or you may use your own blank paper. Upload a scan or photograph of your test paper to the course Canvas page no later than midnight on Friday, November 20, 2020.

**Students enrolled for graduate credit:** work all five problems. Each problem counts 20 points.

**Students enrolled for undergraduate credit:** work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

*On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.*

Name: ____________________ Date: ____________________

1. **25/20 pts.** Let $x[n]$ and $h[n]$ be finite-length discrete-time signals given by

$$\begin{aligned} x[n] &= [9\ 5\ 6\ 5] \\ &= 9\delta[n] + 5\delta[n-1] + 6\delta[n-2] + 5\delta[n-3],\ 0 \le n \le 3, \end{aligned}$$

and

$$\begin{aligned} h[n] &= [1\ \ -2\ \ 2\ \ -1] \\ &= \delta[n] - 2\delta[n-1] + 2\delta[n-2] - \delta[n-3],\ 0 \le n \le 3. \end{aligned}$$

Use the DFT to find the circular convolution $y[n] = x[n] \circledast_4 h[n]$.

$$X[k] = \sum_{n=0}^{3} x[n] W_4^{nk} = 9 + 5W_4^k + 6W_4^{2k} + 5W_4^{3k}$$

$$H[k] = \sum_{n=0}^{3} h[n] W_4^{nk} = 1 - 2W_4^k + 2W_4^{2k} - W_4^{3k}$$

$$Y[k] = X[k]H[k] = (9 + 5W_4^k + 6W_4^{2k} + 5W_4^{3k})(1 - 2W_4^k + 2W_4^{2k} - W_4^{3k})$$

$$\begin{aligned} = 9 &- 18W_4^k + 18W_4^{2k} - 9W_4^{3k} \\ &+ 5W_4^k - 10W_4^{2k} + 10W_4^{3k} - 5W_4^{4k} \\ &+ 6W_4^{2k} - 12W_4^{3k} + 12W_4^{4k} - 6W_4^{5k} \\ &+ 5W_4^{3k} - 10W_4^{4k} + 10W_4^{5k} - 5W_4^{6k} \end{aligned}$$

$$Y[k] = 9 - 13W_4^k + 14W_4^{2k} - 6W_4^{3k} - 3\underbrace{W_4^{4k}}_{=1} + 4\underbrace{W_4^{5k}}_{=W_4^k} - 5\underbrace{W_4^{6k}}_{=W_4^{2k}}$$

$$= (9-3) + (4-13)W_4^k + (14-5)W_4^{2k} - 6W_4^{3k}$$

$$= 6 - 9W_4^k + 9W_4^{2k} - 6W_4^{3k} \quad (*)$$

By definition: $Y[k] = \sum_{n=0}^{3} y[n] W_4^{nk} = y[0] + y[1]W_4^k + y[2]W_4^{2k} + y[3]W_4^{3k}$ $(**)$

Comparing $(*)$ and $(**)$:

$$\boxed{\begin{aligned} y[n] &= [6\ -9\ 9\ -6] \\ &= 6\delta[n] - 9\delta[n-1] + 9\delta[n-2] - 6\delta[n-3],\ 0 \le n \le 3 \end{aligned}}$$

2

More Workspace for Problem 1...

2. **25/20 pts.** A digital filter $G$ has transfer function

$$G(z) = \frac{\left(1 - 2e^{j\pi/3}z^{-1}\right)\left(1 - 2e^{-j\pi/3}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)}{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)\left(1 + 4z^{-1}\right)}, \qquad |z| > 4.$$

Find the transfer function $H(z)$ for a *new* digital filter $H$ such that:

(a) $H(z)$ has the same magnitude response as $G(z)$; i.e., $|H(e^{j\omega})| = |G(e^{j\omega})|\ \forall\, \omega \in \mathbb{R}$.

(b) $H(z)$ is causal, stable, and minimum phase.

→ We know that $G$ is causal... because the ROC of $G(z)$ is exterior.

→ Then $G$ is <u>not</u> <u>stable</u> because of the "bad" pole at $z = -4$, which is <u>outside</u> the unit circle.

→ $G$ is also <u>not</u> <u>minimum</u> <u>phase</u> because of the "bad" zeros at $z = 2e^{j\pi/3}$ and $z = 2e^{-j\pi/3}$, which are also <u>outside</u> the unit circle.

⇒ Use allpass sections to reflect the bad pole and the two bad zeros of $G(z)$ <u>inside</u> the unit circle for $H(z)$.

$$G(z) = \underbrace{\frac{1+\frac{1}{3}z^{-1}}{\left(1+\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}}_{\text{the ``good'' terms}} \cdot \underbrace{\left(1-2e^{j\pi/3}z^{-1}\right)\left(1-2e^{-j\pi/3}z^{-1}\right)}_{\text{the ``bad'' zeros}} \cdot \underbrace{\frac{1}{1+4z^{-1}}}_{\text{the ``bad'' pole}}$$

$$= \frac{1+\frac{1}{3}z^{-1}}{\left(1+\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} \left(1-2e^{j\pi/3}z^{-1}\right) \underbrace{\frac{z^{-1}-2e^{-j\pi/3}}{z^{-1}-2e^{-j\pi/3}}}_{\text{one}} \left(1-2e^{-j\pi/3}z^{-1}\right) \underbrace{\frac{z^{-1}-2e^{j\pi/3}z^{-1}}{z^{-1}-2e^{j\pi/3}z^{-1}}}_{\text{one}}$$

$$\times \frac{1}{1+4z^{-1}} \underbrace{\frac{z^{-1}+4}{z^{-1}+4}}_{\text{one}}$$

$\longrightarrow$

More Workspace for Problem 2...

$$G(z) = \frac{1+\frac{1}{3}z^{-1}}{(1+\frac{1}{4}z^{-1})(1-\frac{1}{4}z^{-1})}\,(z^{-1}-2e^{-j\pi/3})\,\frac{1-2e^{j\pi/3}z^{-1}}{z^{-1}-2e^{-j\pi/3}}$$

$$\times\,(z^{-1}-2e^{j\pi/3})\,\frac{1-2e^{-j\pi/3}z^{-1}}{z^{-1}-2e^{j\pi/3}}\cdot\frac{1}{z^{-1}+4}\;\frac{z^{-1}+4}{1+4z^{-1}}$$

$$= \underbrace{\frac{(1+\frac{1}{3}z^{-1})(z^{-1}-2e^{-j\pi/3})(z^{-1}-2e^{j\pi/3})}{(1+\frac{1}{4}z^{-1})(1-\frac{1}{4}z^{-1})(z^{-1}+4)}}_{H(z)}\cdot\underbrace{\frac{(1-2e^{j\pi/3}z^{-1})(1-2e^{-j\pi/3}z^{-1})(z^{-1}+4)}{(z^{-1}-2e^{-j\pi/3})(z^{-1}-2e^{j\pi/3})(1+4z^{-1})}}_{\text{All Pass}}$$

$$H(z) = \frac{(-2e^{-j\pi/3})(-2e^{j\pi/3})(1+\frac{1}{3}z^{-1})(1-\frac{1}{2}e^{j\pi/3}z^{-1})(1-\frac{1}{2}e^{-j\pi/3}z^{-1})}{4\,(1+\frac{1}{4}z^{-1})(1-\frac{1}{4}z^{-1})(1+\frac{1}{4}z^{-1})}$$

$$= \frac{4(1+\frac{1}{3}z^{-1})(1-\frac{1}{2}e^{j\pi/3}z^{-1})(1-\frac{1}{2}e^{-j\pi/3}z^{-1})}{4(1+\frac{1}{4}z^{-1})^2(1-\frac{1}{4}z^{-1})}$$

poles: $z = -\frac{1}{4}, -\frac{1}{4}, +\frac{1}{4} \Rightarrow$ ROC must be $|z| > \frac{1}{4}$ for causal.

$$\boxed{H(z) = \frac{(1+\frac{1}{3}z^{-1})(1-\frac{1}{2}e^{j\pi/3}z^{-1})(1-\frac{1}{2}e^{-j\pi/3}z^{-1})}{(1+\frac{1}{4}z^{-1})^2(1-\frac{1}{4}z^{-1})} \qquad \text{ROC: } |z| > \frac{1}{4}}$$

3. **25/20 pts.** $H$ is a causal, stable type IV linear phase FIR digital filter with order $N = 5$. The impulse response $h[n]$ is real and $h[0] = 1$.

$H(z)$ has a zero at $z = \frac{1}{2}e^{j\frac{\pi}{3}}$.

Find the transfer function $H(z)$. Be sure to specify the ROC.

→ Because $H$ is Type IV, there must be a zero @ $z = 1$.

→ The zero @ $z = \frac{1}{2}e^{j\pi/3}$ is given.

→ Because $H$ is a linear phase FIR filter, there must also be a zero @ $z = \dfrac{1}{\frac{1}{2}e^{j\pi/3}} = 2e^{-j\pi/3}$.

→ Because $h[n]$ is real, complex zeros must occur in conjugate pairs. So there must also be zeros at

$$z = \left[\tfrac{1}{2}e^{j\pi/3}\right]^* = \tfrac{1}{2}e^{-j\pi/3}$$

$$z = \left[2e^{-j\pi/3}\right]^* = 2e^{j\pi/3}.$$

⇒ We have found 5 zeros so far. Since the order is given as $N=5$, there are only 5 zeros... we have found them all.

→ Then for some constant $K$, we must have that

$$H(z) = K(1-z^{-1})(1-\tfrac{1}{2}e^{j\pi/3}z^{-1})(1-\tfrac{1}{2}e^{-j\pi/3}z^{-1})(1-2e^{j\pi/3}z^{-1})(1-2e^{-j\pi/3}z^{-1})$$

$$= K(1-z^{-1})\left[1-\tfrac{1}{2}(e^{j\pi/3}+e^{-j\pi/3})z^{-1}+\tfrac{1}{4}z^{-2}\right]\left[1-2(e^{j\pi/3}+e^{-j\pi/3})z^{-1}+4z^{-2}\right]$$

$$= K(1-z^{-1})\left[1-\underbrace{\cos(\tfrac{\pi}{3})}_{1/2}z^{-1}+\tfrac{1}{4}z^{-2}\right]\left[1-4\underbrace{\cos(\tfrac{\pi}{3})}_{1/2}z^{-1}+4z^{-2}\right]$$

$$= K(1-z^{-1})(1-\tfrac{1}{2}z^{-1}+\tfrac{1}{4}z^{-2})(1-2z^{-1}+4z^{-2}) \qquad (*)$$

⇒ At this point, it is clear that if we multiply out then the $(z^{-1})^0$ term in $H(z)$ will be $K\cdot 1 \overset{Z}{\longleftrightarrow} K\delta[n] = h[0]\delta[n]$. Since it is given that $h[0]=1$, this means $K=1$. So if this was a timed test, I could stop here by taking $K=1$ in $(*)$ and observing that the ROC must be $|z|>0$ since $H$ is a causal FIR filter. ↦

But let's keep going and show explicitly that this is true:

$$H(z) = (*) = K(1-z^{-1})\Big(1 - 2z^{-1} + 4z^{-2}$$
$$- \tfrac{1}{2}z^{-1} + z^{-2} - 2z^{-3}$$
$$+ \tfrac{1}{4}z^{-2} - \tfrac{1}{2}z^{-3} + z^{-4}\Big)$$

$$= K(1-z^{-1})\left(1 - \tfrac{5}{2}z^{-1} + \tfrac{21}{4}z^{-2} - \tfrac{5}{2}z^{-3} + z^{-4}\right)$$

$$= K\Big(1 - \tfrac{5}{2}z^{-1} + \tfrac{21}{4}z^{-2} - \tfrac{5}{2}z^{-3} + z^{-4}$$
$$- z^{-1} + \tfrac{5}{2}z^{-2} - \tfrac{21}{4}z^{-3} + \tfrac{5}{2}z^{-4} - z^{-5}\Big)$$

$$= K\left(1 - \tfrac{7}{2}z^{-1} + \tfrac{31}{4}z^{-2} - \tfrac{31}{4}z^{-3} + \tfrac{7}{2}z^{-4} - z^{-5}\right) \qquad (**)$$

$\Rightarrow$ Because $H$ is an FIR filter, the poles are all at $z=0$…
i.e., there is a fifth-order pole @ $z=0$.
$\Longrightarrow$ Since $H$ is given to be causal, the ROC must be: $|z|>0$.

$$\mathcal{Z}^{-1}: \quad h[n] = \left[\underset{\uparrow}{K} \;\; -\tfrac{7}{2}K \;\; \tfrac{31}{4}K \;\Big|\; -\tfrac{31}{4}K \;\; \tfrac{7}{2}K \;\; -K\right]$$

$h[0] \Longrightarrow K=1$ (since it is given that $h[0]=1$).

Note: $h[n]$ has odd symmetry, even length, odd order
$\Rightarrow$ Type IV ✓

From $(**)$ with $K=1$, we have:

$$\boxed{\begin{aligned} H(z) &= 1 - \tfrac{7}{2}z^{-1} + \tfrac{31}{4}z^{-2} - \tfrac{31}{4}z^{-3} + \tfrac{7}{2}z^{-4} - z^{-5} \\ &= 1 - 3.5z^{-1} + 7.75z^{-2} - 7.75z^{-3} + 3.5z^{-4} - z^{-5}, \\ &\text{ROC: } |z|>0 \end{aligned}}$$

→ Numbers in boxes like $\boxed{05}$ are calculator registers.

4. **25/20 pts.** Design a causal, stable "simple" second-order IIR digital bandstop filter with a notch frequency of $\omega_0 = \frac{\pi}{3}$ rad/sample and a quality of $Q = 4$.

Give the transfer function $H(z)$ and be sure to specify the ROC.

**Hint:** "simple" digital filters are discussed on pages 7.52–7.59 of the course lecture notes.

Notes p. 7.59: $\omega_0 = \arccos\beta \Rightarrow \beta = \cos\omega_0 = \cos\frac{\pi}{3} = \frac{1}{2}$ : $\boxed{\beta = \frac{1}{2}}$

$$Q = \frac{\omega_0}{B_W} \Rightarrow B_W = \frac{\omega_0}{Q} = \frac{\pi/3}{4} = \frac{\pi}{12}$$

$$B_W = \arccos\left(\frac{2\alpha}{1+\alpha^2}\right) \Rightarrow \underbrace{\frac{2\alpha}{1+\alpha^2} = \cos B_W}_{} = \cos\frac{\pi}{12} = \frac{1}{4}(\sqrt{6}+\sqrt{2})$$

$$= 0.965926 \quad \boxed{05}$$

$$2\alpha = \cos B_W + \alpha^2 \cos B_W$$

$$[\cos B_W]\alpha^2 - 2\alpha + \cos B_W = 0$$

Quadratic Formula: $a = \cos B_W \quad b = -2 \quad c = \cos B_W$

$$\alpha = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4(\cos B_W)^2}}{2\cos B_W}$$

$$= \frac{2 \pm \sqrt{0.267949}}{2(0.965926)} = \frac{2 \pm 0.517638 \; \boxed{06}}{1.93185 \; \boxed{07}}$$

$\boxed{05}$

$$\Rightarrow \alpha = 1.30323 \quad \text{or} \quad \alpha = 0.767327 \; \boxed{08}$$

→ Note that the bandstop filter on Notes p. 7.59 has the <u>same</u> denominator as the bandpass filter on p. 7.58. Only the numerators are different. So the poles of the bandstop filter are the <u>same</u> as the poles of the bandpass filter.

Notes p. 7.58: For stability, we must choose $|\alpha| < 1$.

$$\Rightarrow \boxed{\alpha = 0.767327} \; \boxed{08}$$



→

Also from Notes p. 7.58, the poles of the bandpass filter, and so also the poles of the bandstop filter, are at $z = re^{\pm j\phi}$, where $r = \sqrt{\alpha}$ and $\phi = \arccos\left[\frac{\beta(1+\alpha)}{2\sqrt{\alpha}}\right]$.

$\Rightarrow$ There are two poles, and they both have magnitude $r = \sqrt{\alpha} = 0.875972$.

$\Rightarrow$ For the bandstop filter to be <u>causal</u>, the ROC must be:

$|z| > \sqrt{\alpha} = 0.875972$.

Notes p. 7.59:

$$H(z) = \frac{1+\alpha}{2} \; \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

$$= \frac{1+\alpha}{2} \; \frac{1 - z^{-1} + z^{-2}}{1 - \frac{1}{2}(1+\alpha)z^{-1} + \alpha z^{-2}}$$

$$= \underset{\boxed{09}}{0.883663} \; \frac{1 - z^{-1} + z^{-2}}{1 - 0.883663 z^{-1} + 0.767327 z^{-2}}$$

$$\boxed{H(z) = \frac{0.883663(1 - z^{-1} + z^{-2})}{1 - 0.883663 z^{-1} + 0.767327 z^{-2}} \qquad \text{ROC: } |z| > 0.875972}$$

$\rightarrow$ Using matlab, you can verify that the frequency response $H(e^{j\omega})$ has the expected behavior.

Numbers in boxes like this $\boxed{00}$ are calculator registers.

5. **25/20 pts.** Design an analog Type 2 Chebyshev low pass filter to meet the following analog design specification:

$\frac{1}{A} = \frac{1}{3}$

$A = 3$

| | |
|---|---|
| passband edge freq. | $\Omega_p = 5\pi$ rad/sec |
| stopband edge freq. | $\Omega_s = 10\pi$ rad/sec |
| min. stopband attenuation | $1/A = \frac{1}{3}$ |
| max. passband attenuation | $1/\sqrt{1+\varepsilon^2} = 0.9$ |

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{9}{10}$$
$$\sqrt{1+\varepsilon^2} = \frac{10}{9}$$
$$1+\varepsilon^2 = \frac{100}{81}$$
$$\varepsilon^2 = \frac{100}{81} - 1 = \frac{100-81}{81} = 19/81$$
$$\varepsilon = \sqrt{19/81}$$
$$\varepsilon = 0.484322 \quad \boxed{05}$$

Formulas are on Notes pages A.8 through A.9

Give the analog filter transfer function $H_a(s)$.

**Hint:** Make sure that your calculator is set for **radians** and **not** degrees!

$$N = \left\lceil \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\cosh^{-1}(5.83997)}{\cosh^{-1}(2)} \right\rceil = \left\lceil \frac{2.45046}{1.31696} \right\rceil$$
$$= \lceil 1.86070 \rceil = 2 \longrightarrow \boxed{N=2}$$

$$z_1 = \frac{j\Omega_s}{\cos\frac{\pi}{4}} = \frac{j10\pi}{\frac{1}{2}\sqrt{2}} = j\,44.4288 \quad \boxed{06}$$

$$z_2 = \frac{j\Omega_s}{\cos\frac{3\pi}{4}} = \frac{j10\pi}{-\frac{1}{2}\sqrt{2}} = -j44.4288 = z_1^*$$

$$\gamma = (A+\sqrt{A^2-1})^{\frac{1}{N}} = (3+\sqrt{8})^{1/2} = 2.41421 \quad \boxed{07}$$

$$\xi = \frac{\gamma^2+1}{2\gamma} = 1.41421 \quad \boxed{08} \qquad \zeta = \frac{\gamma^2-1}{2\gamma} = 1$$

$$\alpha_1 = -\Omega_p \zeta \sin\frac{\pi}{4} = -11.1072 \quad \boxed{09}$$

$$\alpha_2 = -\Omega_p \zeta \sin\frac{3\pi}{4} = -11.1072 = \alpha_1$$

$$\beta_1 = \Omega_p \xi \cos\frac{\pi}{4} = 15.7080 \quad \boxed{10}$$

$$\beta_2 = \Omega_p \xi \cos\frac{3\pi}{4} = -15.7080 = -\beta_1$$

$$\sigma_1 = \frac{\Omega_s\alpha_1}{\alpha_1^2+\beta_1^2} = -0.942809 \quad \boxed{11}$$

$$\sigma_2 = \frac{\Omega_s\alpha_2}{\alpha_2^2+\beta_2^2} = -0.942809 = \sigma_1$$

$$\Omega_1 = -\frac{\Omega_s\beta_1}{\alpha_1^2+\beta_1^2} = -1.33333 \quad \boxed{12}$$

$$\Omega_2 = \frac{-\Omega_s\beta_2}{\alpha_2^2+\beta_2^2} = 1.33333 = -\Omega_1$$



$\longrightarrow$

$$p_1 = \sigma_1 + j\Omega_1 = \boxed{11}\,-0.942809 - j\,\boxed{12}\,1.33333$$

More Workspace for Problem 5...

$$p_2 = \sigma_2 + j\Omega_2 = \sigma_1 - j\Omega_1 = -0.942809 + j\,1.33333 = p_1^*.$$

$$H_a(s) = C_0 \frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)} = C_0 \frac{(s-z_1)(s-z_1^*)}{(s-p_1)(s-p_1^*)}$$

$$\left\{ \begin{array}{l} H_a(0) = 1 = C_0 \dfrac{z_1 z_1^*}{p_1 p_1^*} = C_0 \dfrac{|z_1|^2}{|p_1|^2} \\ C_0 = \dfrac{|p_1|^2}{|z_1|^2} = \dfrac{\boxed{13}\,2.66667}{1973.92} = \boxed{14}\,1.35095 \times 10^{-3} \end{array} \right.$$

$$H_a(s) = \frac{|p_1|^2}{|z_1|^2} \frac{(s-z_1)(s-z_1^*)}{(s-p_1)(s-p_1^*)} = \frac{|p_1|^2}{|z_1|^2} \frac{s^2 - (z_1+z_1^*)s + z_1 z_1^*}{s^2 - (p_1+p_1^*)s + p_1 p_1^*}$$

$$= \frac{|p_1|^2}{|z_1|^2} \frac{s^2 - 2\mathrm{Re}\{z_1\}s + |z_1|^2}{s^2 - 2\mathrm{Re}\{p_1\}s + |p_1|^2} = \frac{|p_1|^2}{|z_1|^2} \frac{s^2 - 0s + |z_1|^2}{s^2 - 2\sigma_1 s + |p_1|^2}$$

$$= \frac{\frac{|p_1|^2}{|z_1|^2} s^2 + |p_1|^2}{s^2 - 2\sigma_1 s + |p_1|^2}$$

$$\boxed{H_a(s) = \frac{1.35095 \times 10^{-3} s^2 + 2.66667}{s^2 + 1.885625 + 2.66667}}$$

More Workspace for Problem 5...