ECE 4213/5213
Test 2

Monday, November 29, 2021

Fall 2021 — Name: SOLUTION

Dr. Havlicek — Student Num:

Directions: This test is open book and open notes. You may also use your calculator and the course formula sheet. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ______
2. (25/20) ______
3. (25/20) ______
4. (25/20) ______
5. (25/20) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ______ Date: ______

1. **25/20 pts.** Let $x[n]$ and $h[n]$ be finite-length discrete-time signals given by

$$\begin{aligned} x[n] &= [3\ 1\ 4\ 1]. \\ &= 3\delta[n] + \delta[n-1] + 4\delta[n-2] + \delta[n-3],\ 0 \le n \le 3, \end{aligned}$$

and

$$\begin{aligned} h[n] &= [-2 \quad 1 \quad -4 \quad 5] \\ &= -2\delta[n] + \delta[n-1] - 4\delta[n-2] + 5\delta[n-3],\ 0 \le n \le 3. \end{aligned}$$

Use the DFT to find the circular convolution $y[n] = x[n] \circledast_4 h[n]$.

$$X[k] = \sum_{n=0}^{3} x[n] W_4^{nk} \qquad\qquad H[k] = \sum_{n=0}^{3} h[n] W_4^{nk}$$

$$= 3 + W_4^{k} + 4W_4^{2k} + W_4^{3k} \qquad\qquad = -2 + W_4^{k} - 4W_4^{2k} + 5W_4^{3k}$$

$$Y[k] = X[k]H[k] = (3 + W_4^{k} + 4W_4^{2k} + W_4^{3k})(-2 + W_4^{k} - 4W_4^{2k} + 5W_4^{3k})$$

$$\begin{array}{rrrrrrrr} = -6 & + 3W_4^{k} & - 12W_4^{2k} & + 15W_4^{3k} & & & & \\ & - 2W_4^{k} & + W_4^{2k} & - 4W_4^{3k} & + 5W_4^{4k} & & & \\ & & - 8W_4^{2k} & + 4W_4^{3k} & - 16W_4^{4k} & + 20W_4^{5k} & & \\ & & & - 2W_4^{3k} & + W_4^{4k} & - 4W_4^{5k} & + 5W_4^{6k} \end{array}$$

$$Y[k] = -6 + W_4^{k} - 19W_4^{2k} + 13W_4^{3k} - 10\underbrace{W_4^{4k}}_{1} + 16\underbrace{W_4^{5k}}_{W_4^{k}} + 5\underbrace{W_4^{6k}}_{W_4^{2k}}.$$

$$= (-6-10) + (1+16)W_4^{k} + (5-19)W_4^{2k} + 13W_4^{3k}$$

$$Y[k] = -16 + 17W_4^{k} - 14W_4^{2k} + 13W_4^{3k} \quad (*)$$

$$= \sum_{n=0}^{3} y[n] W_4^{k} = y[0] + y[1]W_4^{k} + y[2]W_4^{2k} + y[3]W_4^{3k} \quad (**)$$

Comparing $(*)$ and $(**)$, we have:

$$\boxed{\begin{aligned} y[n] &= [-16\ 17\ -14\ 13]_2 \\ &= -16\delta[n] + 17\delta[n-1] - 14\delta[n-2] + 13\delta[n-3],\ 0 \le n \le 3 \end{aligned}}$$

More Workspace for Problem 1...

2. **25/20 pts.** A digital filter F has transfer function

$$F(z) = \frac{\left(1-\frac{2}{3}z^{-1}\right)\left(1+\frac{2}{3}z^{-1}\right)\left(1-\frac{5}{2}z^{-1}\right)}{\left(1-\frac{3}{2}e^{\jmath\pi/6}z^{-1}\right)\left(1-\frac{3}{2}e^{-\jmath\pi/6}z^{-1}\right)\left(1-\frac{1}{6}z^{-1}\right)}, \qquad |z| > \frac{3}{2}.$$

Find the transfer function $H(z)$ for a *new* digital filter H such that·

(a) $H(z)$ has the same magnitude response as $F(z)$; i.e., $|H(e^{\jmath\omega})| = |F(e^{\jmath\omega})|\ \forall\, \omega \in \mathbb{R}$.

(b) $H(z)$ is causal, stable, and minimum phase

- $F(z)$ is not stable because of the bad poles at $z = \frac{3}{2}e^{\pm j\pi/6}$, which are outside of the unit circle.
- $F(z)$ is not minimum phase because of the bad zero at $z = 5/2$, which is also outside of the unit circle.
- For $H(z)$, we must use allpass sections to reflect the bad zero and bad poles inside the unit circle.

$$F(z) = \underbrace{\frac{(1-\frac{2}{3}z^{-1})(1+\frac{2}{3}z^{-1})}{(1-\frac{1}{6}z^{-1})}}_{\text{the good terms}} \cdot \underbrace{(1-\tfrac{5}{2}z^{-1})}_{\text{the bad zero}} \cdot \underbrace{\frac{1}{(1-\frac{3}{2}e^{j\pi/6}z^{-1})(1-\frac{3}{2}e^{-j\pi/6}z^{-1})}}_{\text{the bad poles}}$$

$$= \frac{(1-\frac{2}{3}z^{-1})(1+\frac{2}{3}z^{-1})}{(1-\frac{1}{6}z^{-1})} \cdot (1-\tfrac{5}{2}z^{-1}) \cdot \underbrace{\frac{z^{-1}-5/2}{z^{-1}-5/2}}_{\text{one}}$$

$$\cdot \frac{1}{(1-\frac{3}{2}e^{j\pi/6}z^{-1})} \cdot \underbrace{\frac{z^{-1}-\frac{3}{2}e^{-j\pi/6}}{z^{-1}-\frac{3}{2}e^{-j\pi/6}}}_{\text{one}}$$

$$\cdot \frac{1}{(1-\frac{3}{2}e^{-j\pi/6}z^{-1})} \cdot \underbrace{\frac{z^{-1}-\frac{3}{2}e^{j\pi/6}}{z^{-1}-\frac{3}{2}e^{j\pi/6}}}_{\text{one}}$$

$\longrightarrow$

More Workspace for Problem 2...

$$F(z) = \underbrace{\frac{(1-\frac{2}{3}z^{-1})(1+\frac{2}{3}z^{-1})(z^{-1}-\frac{5}{2})}{(1-\frac{1}{6}z^{-1})(z^{-1}-\frac{3}{2}e^{-j\pi/6})(z^{-1}-\frac{3}{2}e^{j\pi/6})}}_{H(z)}$$

$$\times \underbrace{\frac{(1-\frac{5}{2}z^{-1})(z^{-1}-\frac{3}{2}e^{-j\pi/6})(z^{-1}-\frac{3}{2}e^{j\pi/6})}{(z^{-1}-\frac{5}{2})(1-\frac{3}{2}e^{j\pi/6}z^{-1})(1-\frac{3}{2}e^{-j\pi/6}z^{-1})}}_{\text{all pass}}$$

$$H(z) = \frac{(1-\frac{2}{3}z^{-1})(1+\frac{2}{3}z^{-1})(-\frac{5}{2})(1-\frac{2}{5}z^{-1})}{(1-\frac{1}{6}z^{-1})(-\frac{3}{2}e^{-j\pi/6})(1-\frac{2}{3}e^{j\pi/6}z^{-1})(-\frac{3}{2}e^{j\pi/6})(1-\frac{2}{3}e^{-j\pi/6}z^{-1})}$$

$$= \frac{-5/2\,(1-\frac{2}{3}z^{-1})(1+\frac{2}{3}z^{-1})(1-\frac{2}{5}z^{-1})}{\frac{9}{4}(1-\frac{1}{6}z^{-1})(1-\frac{2}{3}e^{j\pi/6}z^{-1})(1-\frac{2}{3}e^{-j\pi/6}z^{-1})}$$

$$= -\frac{10(1-\frac{2}{3}z^{-1})(1+\frac{2}{3}z^{-1})(1-\frac{2}{5}z^{-1})}{9(1-\frac{1}{6}z^{-1})(1-\frac{2}{3}e^{j\pi/6}z^{-1})(1-\frac{2}{3}e^{-j\pi/6}z^{-1})}$$

poles: $z = \frac{1}{6}, \frac{2}{3}e^{j\pi/6}, \frac{2}{3}e^{-j\pi/6}$

ROC must be $|z| > 2/3$ for causal

$$\boxed{H(z) = -\frac{10(1-\frac{2}{3}z^{-1})(1+\frac{2}{3}z^{-1})(1-\frac{2}{5}z^{-1})}{9(1-\frac{1}{6}z^{-1})(1-\frac{2}{3}e^{j\pi/6}z^{-1})(1-\frac{2}{3}e^{-j\pi/6}z^{-1})} \quad \text{ROC: } |z| > \frac{2}{3}}$$

3. **25/20 pts.** H is a causal, stable fifth-order **low pass** linear phase FIR digital filter. The impulse response $h[n]$ is real.

$H(z)$ has a zero at $z = \frac{3}{2}e^{j\pi/4}$.

Give a pole-zero plot for $H(z)$.

Hint 1: You do *not* have to find the transfer function $H(z)$.

Hint 2: It is an FIR filter – don't forget about the poles when you make your plot!

NOTES p. 7.51: because H is a low pass linear phase FIR filter, it must be type I or type II.

NOTES p. 7.39: because the order is odd, H must be type II.

NOTES p. 7.48: A type II linear phase FIR filter must have a zero @ $z=-1$.

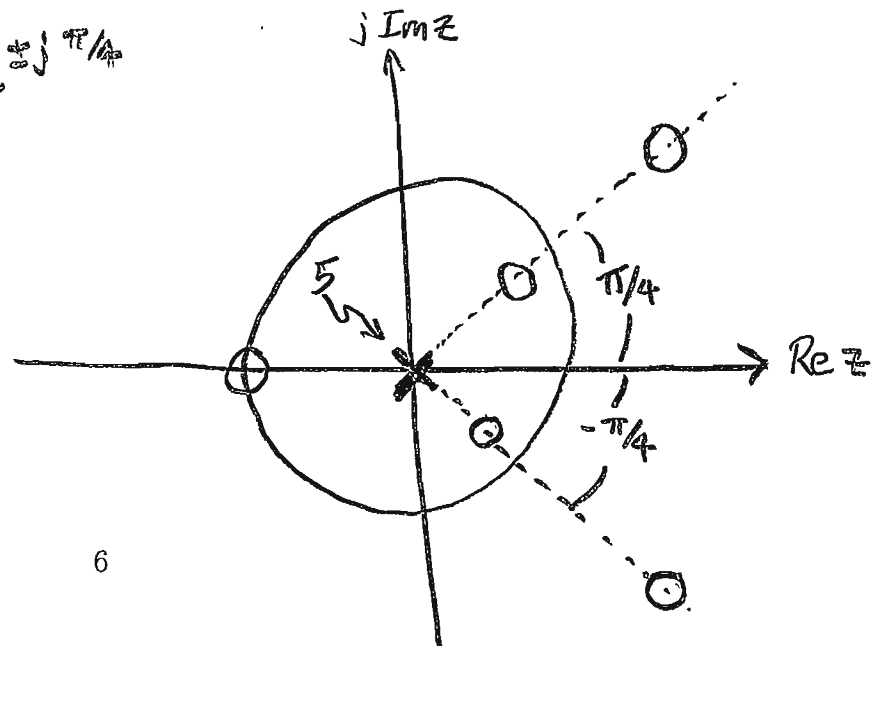
$\Rightarrow$ It is given that $H(z)$ has a zero @ $z = \frac{3}{2}e^{j\pi/4}$.

- Because $h[n] \in \mathbb{R}$, there must also be a zero @ $z = \frac{3}{2}e^{-j\pi/4}$
- Because H is a linear phase FIR filter, there must also be zeros @ $z = \frac{2}{3}e^{j\pi/4}$ and $z = \frac{2}{3}e^{-j\pi/4}$.

NOTES p. 3.122: Because H is an FIR filter, there must also be a fifth-order pole @ $z=0$.

Poles: $z=0$ (5th order)

Zeros: $z=-1$, $\frac{3}{2}e^{\pm j\pi/4}$, $\frac{2}{3}e^{\pm j\pi/4}$



4. **25/20 pts.** Design a causal, stable "simple" second-order IIR digital bandpass filter with a passband center frequency of $\omega_0 = \frac{\pi}{3}$ rad/sample and a quality of $Q = 1$.

Give the transfer function $H(z)$ and be sure to specify the ROC.

Hint: "simple" digital filters are discussed on pages 7.52–7.59 of the course lecture notes.

$$\omega_0 = \arccos\beta \longrightarrow \beta = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$Q = \frac{\omega_0}{B_W} \longrightarrow B_W = \frac{\omega_0}{Q} = \omega_0 = \pi/3$$

$$B_W = \frac{\pi}{3} = \arccos\left(\frac{2\alpha}{1+\alpha^2}\right)$$

$$\cos\frac{\pi}{3} = \frac{1}{2} = \frac{2\alpha}{1+\alpha^2} \longrightarrow 4\alpha = 1+\alpha^2 \longrightarrow \alpha^2 - 4\alpha + 1 = 0$$

$$a=1,\ b=-4,\ c=1$$

$$\alpha = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \frac{1}{2}\sqrt{12}$$

$$= 2 \pm \sqrt{3}$$

$$\alpha = 3.73205 \ \underline{\text{or}}\ \alpha = 0.267949$$

→ For stability, we must have $|\alpha| < 1$: $\alpha = 0.267949$

→ The poles are at $re^{\pm j\phi}$, where $r = \sqrt{\alpha} = 0.517638$.

→ For causal + stable, the ROC must be $|z| > 0.517638$

$$H(z) = \frac{(1-\alpha)}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

$$\boxed{H(z) = \frac{0.366025(1-z^{-2})}{1-0.633975z^{-1}+0.267949z^{-2}}, \quad |z| > 0.517638}$$

More Workspace for Problem 4...

The numbers in boxes like this $\boxed{00}$ are calculator registers.

5. **25/20 pts.** Design an analog Type 1 Chebyshev low pass filter to meet the following analog design specification:

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{9}{10} \qquad 1+\varepsilon^2 = \frac{100}{81} \qquad \varepsilon^2 = \frac{100}{81} - 1 = \frac{100-81}{81} = 19/81 \qquad \varepsilon = \frac{\sqrt{19}}{9}$$

passband edge freq.	$\Omega_p = 2000\pi$ rad/sec
stopband edge freq.	$\Omega_s = 8000\pi$ rad/sec
min. stopband attenuation	$1/A = 0.01$
max. passband attenuation	$1/\sqrt{1+\varepsilon^2} = 0.90$

$$\frac{1}{A} = \frac{1}{100} \qquad A = 100$$

Give the analog filter transfer function $H_a(s)$.

Hint: Make sure that your calculator is set for **radians** and **not** degrees!

$$\overset{\boxed{05}}{\varepsilon = 0.484322}$$

$$N = \left\lceil \frac{\cosh^{-1}\left(\sqrt{A^2-1}/\varepsilon\right)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\cosh^{-1}\left(\sqrt{9{,}999}/\varepsilon\right)}{\cosh^{-1}(4)} \right\rceil = \left\lceil \frac{\cosh^{-1}(\overset{\boxed{06}}{206.464})}{\cosh^{-1}(4)} \right\rceil$$

$$= \left\lceil \frac{\overset{\boxed{07}}{6.02327}}{\underset{\boxed{08}}{2.06344}} \right\rceil = \lceil 2.91905 \rceil = 3$$

$$\gamma = \left(\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)^{\frac{1}{N}} = \left(\frac{1+10/9}{\sqrt{19}/9}\right)^{1/3} \doteq (4.35890)^{1/3} = \overset{\boxed{09}}{1.63352}$$

$$\zeta = \frac{\gamma^2+1}{2\gamma} = \overset{\boxed{10}}{1.12285} \qquad \xi = \frac{\gamma^2-1}{2\gamma} = \overset{\boxed{11}}{0.510675}$$

$$\sigma_1 = -\Omega_p \xi \sin\left[\frac{\pi}{6}\right] = \overset{\boxed{12}}{-1{,}604.33}$$

$$\sigma_2 = -\Omega_p \xi \sin\left[\frac{\pi}{2}\right] = \overset{\boxed{13}}{-3{,}208.67}$$

$$\sigma_3 = -\Omega_p \xi \sin\left[\frac{5\pi}{6}\right] = \overset{\boxed{12}}{-1{,}604.33} = \sigma_1$$

$$\Omega_1 = \Omega_p \zeta \cos\left[\frac{\pi}{6}\right] = \overset{\boxed{14}}{6{,}109.87}$$

$$\Omega_2 = \Omega_p \zeta \cos\left[\frac{\pi}{2}\right] = 0$$

$$\Omega_3 = \Omega_p \zeta \cos\left[\frac{5\pi}{6}\right] = \overset{\boxed{15}}{-6{,}109.87} = -\Omega_1$$

$$p_1 = \sigma_1 + j\Omega_1 = \overset{\boxed{12}}{-1{,}604.33} + j\,\overset{\boxed{14}}{6{,}109.87}$$

$$p_2 = \sigma_2 + j\Omega_2 = \overset{\boxed{13}}{-3{,}208.67}$$

$$p_3 = \sigma_3 + j\Omega_3 = \sigma_1 - j\Omega_1 = \overset{\boxed{12}}{-1{,}604.33} - j\,\overset{\boxed{15}}{6{,}109.87} = p_1^*$$

N odd: $C_0 = 1$ $\longrightarrow$

More Workspace for Problem 5...

$$H_a(s) = C_0 \prod_{\ell=1}^{3} \frac{-p_\ell}{s-p_\ell} = \frac{(-p_1)(-p_2)(-p_3)}{(s-p_1)(s-p_2)(s-p_3)}$$

$$= \frac{-p_1 p_1^* p_2}{(s-p_1)(s-p_1^*)(s-p_2)} = \frac{-|p_1|^2 p_2}{(s^2 - 2\mathrm{Re}[p_1]s + |p_1|^2)(s-p_2)}$$

$$= \frac{-(\sigma_1^2+\Omega_1^2)\sigma_2}{[s^2 - 2\sigma_1 s + (\sigma_1^2+\Omega_1^2)](s-\sigma_2)}$$

$$= \frac{-(\sigma_1^2+\Omega_1^2)\sigma_2}{s^3 - \sigma_2 s^2 - 2\sigma_1 s^2 + 2\sigma_1\sigma_2 s + (\sigma_1^2+\Omega_1^2)s - (\sigma_1^2+\Omega_1^2)\sigma_2}$$

$$= \frac{-(\sigma_1^2+\Omega_1^2)\sigma_2}{s^3 - (2\sigma_1+\sigma_2)s^2 + [2\sigma_1\sigma_2 + (\sigma_1^2+\Omega_1^2)]s - (\sigma_1^2+\Omega_1^2)\sigma_2}$$

$$\boxed{H_a(s) = \frac{128.040\times 10^9}{s^3 + 6{,}417.34 s^2 + 50.1999\times 10^6 s + 128.040\times 10^9}}$$

More Workspace for Problem 5...