ECE 4213/5213
Test 2

Wednesday, November 30, 2022

Fall 2022 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: This test is open book and open notes. You may also use your calculator and the course formula sheet. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ______
2. (25/20) ______
3. (25/20) ______
4. (25/20) ______
5. (25/20) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ______ Date: ______

1. **25/20 pts.** Let $x[n]$ and $h[n]$ be finite-length discrete-time signals given by

$$\begin{aligned} x[n] &= [1\ 2\ 3\ 4] \\ &= \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3],\ 0 \le n \le 3, \end{aligned}$$

and

$$\begin{aligned} h[n] &= [-1\quad 2\ \ -1] \\ &= -\delta[n] + 2\delta[n-1] - \delta[n-2],\ 0 \le n \le 2. \end{aligned}$$

Use the DFT to find the *circular* convolution $y[n] = x[n] \circledast h[n]$.

Length $\{x[n]\} = 4$. Length $\{h[n]\} = 3$. $\Rightarrow$ Zero pad to <u>length $N=4$</u>.

$\Rightarrow$ Will be using $W_4 = e^{-j2\pi/4}$

$$x[n] = [1\ 2\ 3\ 4] \Rightarrow X[k] = \sum_{n=0}^{3} x[n] W_4^{nk} = 1 + 2W_4^{k} + 3W_4^{2k} + 4W_4^{3k}$$

$$h_4[n] = [-1\ 2\ -1\ 0] \Rightarrow H[k] = \sum_{n=0}^{3} h_4[n] W_4^{nk} = -1 + 2W_4^{k} - W_4^{2k}$$

$$\begin{aligned} Y[k] = H[k]X[k] &= (-1 + 2W_4^{k} - W_4^{2k})(1 + 2W_4^{k} + 3W_4^{2k} + 4W_4^{3k}) \\ &= -1 - 2W_4^{k} - 3W_4^{2k} - 4W_4^{3k} \\ &\quad + 2W_4^{k} + 4W_4^{2k} + 6W_4^{3k} + 8W_4^{4k} \\ &\quad - W_4^{2k} - 2W_4^{3k} - 3W_4^{4k} - 4W_4^{5k} \end{aligned}$$

$$= -1 + 0W_4^{k} + 0W_4^{2k} + 0W_4^{3k} + \underbrace{5W_4^{4k} - 4W_4^{5k}}_{5W_4^{0} - 4W_4^{k}}$$

$$Y[k] = 4 - 4W_4^{k}$$

Comparing to $Y[k] \equiv \sum_{n=0}^{3} y[n] W_4^{nk} = y[0] + y[1]W_4^{k} + y[2]W_4^{2k} + y[3]W_4^{3k}$,

we get imediately:

$$\boxed{\begin{aligned} y[n] &= [4\ -4\ \ 0\ \ 0] \\ &= 4\delta[n] - 4\delta[n-1],\quad 0 \le n \le 3 \end{aligned}}$$

Doing the multiplication the "other way", we get

$$Y[k] = X[k]H[k] = (1+2W_4^k+3W_4^{2k}+4W_4^{3k})(-1+2W_4^k-W_4^{2k})$$

$$\begin{aligned} = & -1 + 2W_4^k - W_4^{2k} \\ & -2W_4^k + 4W_4^{2k} - 2W_4^{3k} \\ & -3W_4^{2k} + 6W_4^{3k} - 3W_4^{4k} \\ & -4W_4^{3k} + 8W_4^{4k} - 4W_4^{5k} \end{aligned}$$

$$= -1 + 0W_4^k + 0W_4^{2k} + 0W_4^{3k} + \underbrace{5W_4^{4k} - 4W_4^{5k}}_{5W_4^0 - 4W_4^k}$$

$Y[k] = 4 - 4W_4^k$ (same as the other way)

By inspection:

$$\boxed{\begin{aligned} y[n] &= [4 \quad -4 \quad 0 \quad 0] \\ &= 4\delta[n] - 4\delta[n-1], \quad 0 \le n \le 3 \end{aligned}}$$

$\rightarrow$ agrees with the answer on page 2.

2. **25/20 pts.** A digital filter F has transfer function

$$F(z) = \frac{\left(1 - 7e^{j\pi/7}z^{-1}\right)\left(1 - 7e^{-j\pi/7}z^{-1}\right)\left(1 + \frac{1}{7}z^{-1}\right)}{\left(1 - \frac{1}{5}e^{j\pi/5}z^{-1}\right)\left(1 - \frac{1}{5}e^{-j\pi/5}z^{-1}\right)\left(1 + \frac{1}{5}z^{-1}\right)}, \qquad |z| > \frac{1}{5}.$$

(a) **19/15 pts.** Find the transfer function $H(z)$ for a *new* digital filter H such that:

i. $H(z)$ has the same magnitude response as $F(z)$; i.e., $|H(e^{j\omega})| = |F(e^{j\omega})|\ \forall\ \omega \in \mathbb{R}$.

ii. $H(z)$ is causal, stable, and minimum phase.

- For causal, stable, and minimum phase, we need all poles and all zeros to be inside the unit circle.
  - There are three poles, and they all have magnitude $1/5$, which is inside the unit circle. So the poles do not need to be changed.
  - There are two bad zeros outside the unit circle at $z = 7e^{\pm j\pi/7}$. These need to be reflected to new locations inside the unit circle.

$$F(z) = \underbrace{\frac{\left(1+\frac{1}{7}z^{-1}\right)}{\text{denom}}}_{\text{Good Part}}\ \underbrace{\left(1-7e^{j\pi/7}z^{-1}\right)}_{\text{Bad Zero}}\underbrace{\left(1-7e^{-j\pi/7}z^{-1}\right)}_{\text{Bad Zero}}$$

$$= \underbrace{\frac{\left(1+\frac{1}{7}z^{-1}\right)}{\text{denom}}}_{\text{Good Part}}\ \underbrace{\left(1-7e^{j\pi/7}z^{-1}\right)}_{\text{Bad Zero}}\ \underbrace{\frac{\left(z^{-1}-7e^{-j\pi/7}\right)}{\left(z^{-1}-7e^{-j\pi/7}\right)}}_{\text{one}}\ \underbrace{\left(1-7e^{-j\pi/7}z^{-1}\right)}_{\text{Bad Zero}}\ \underbrace{\frac{\left(z^{-1}-7e^{j\pi/7}\right)}{\left(z^{-1}-7e^{j\pi/7}\right)}}_{\text{one}}$$

$$= \underbrace{\frac{\left(1+\frac{1}{7}z^{-1}\right)\left(z^{-1}-7e^{-j\pi/7}\right)\left(z^{-1}-7e^{j\pi/7}\right)}{\text{denom}}}_{H(z)} \cdot \underbrace{\frac{\left(1-7e^{j\pi/7}z^{-1}\right)\left(1-7e^{-j\pi/7}z^{-1}\right)}{\left(z^{-1}-7e^{-j\pi/7}\right)\left(z^{-1}-7e^{j\pi/7}\right)}}_{\text{2nd-order All Pass}}$$

Problem 2, cont...

$$H(z) = \frac{(1+\frac{1}{7}z^{-1})(z^{-1}-7e^{-j\pi/7})(z^{-1}-7e^{j\pi/7})}{(1+\frac{1}{5}z^{-1})(1-\frac{1}{5}e^{j\pi/5}z^{-1})(1-\frac{1}{5}e^{-j\pi/5}z^{-1})}$$

$$= \frac{(1+\frac{1}{7}z^{-1})(-7e^{-j\pi/7})(1-\frac{1}{7}e^{j\pi/7}z^{-1})(-7e^{j\pi/7})(1-\frac{1}{7}e^{-j\pi/7}z^{-1})}{(1+\frac{1}{5}z^{-1})(1-\frac{1}{5}e^{j\pi/5}z^{-1})(1-\frac{1}{5}e^{-j\pi/5}z^{-1})}$$

$$\boxed{H(z) = \frac{49(1-\frac{1}{7}e^{j\pi/7}z^{-1})(1-\frac{1}{7}e^{-j\pi/7}z^{-1})(1+\frac{1}{7}z^{-1})}{(1-\frac{1}{5}e^{j\pi/5}z^{-1})(1-\frac{1}{5}e^{-j\pi/5}z^{-1})(1+\frac{1}{5}z^{-1})}, \quad |z| > \frac{1}{5}}$$

(b) **6/5 pts.** List the poles and zeros of your *new* transfer function $H(z)$ and specify the region of convergence. Briefly explain how you know that $H(z)$ is causal, stable, and minimum phase.

Hint: a pole-zero plot is *not* required for this part.

Poles: $z = \frac{1}{5}e^{\pm j\pi/5}, -\frac{1}{5}$; Zeros: $z = \frac{1}{7}e^{\pm j\pi/7}, -\frac{1}{7}$

ROC: $|z| > 1/5$

Since the ROC is exterior, $h[n]$ must be right-sided. Moreover, since there are no positive powers of z in $H(z)$, $h[n]$ must be right sided and zero $\forall n < 0$. This establishes causality. Stable and minimum phase follow because all poles and zeros are inside the unit circle.

3. **25/20 pts.** H is a causal, stable fifth-order Type IV linear phase FIR digital filter. The impulse response $h[n]$ is real.

$H(z)$ has a zero at $z = \frac{1}{2}e^{-j\pi/4}$.

(a) **13/10 pts.** Give a pole-zero plot for $H(z)$.

Hint 1: You do *not* have to find the transfer function $H(z)$.

Hint 2: It is an FIR filter – don't forget about the poles when you make your plot!

→ A type IV linear phase FIR filter must have a zero @ $z = \underline{1}$. (notes p. 7.51)

→ There is a given zero @ $z = \frac{1}{2}e^{-j\pi/4}$

→ because $h[n]$ is real, there must be another zero @ $z = \frac{1}{2}e^{+j\pi/4}$

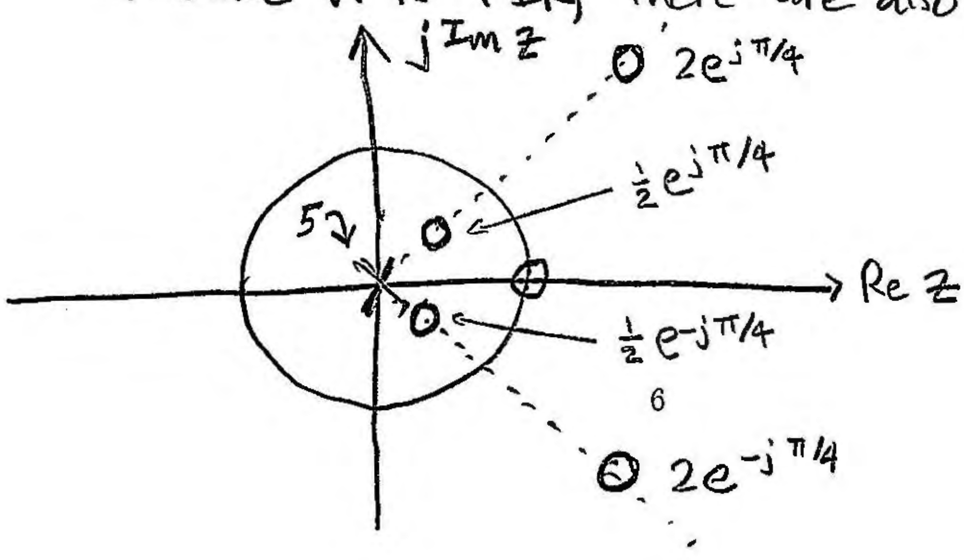
→ because H is a linear phase FIR filter, there must be another zero at $z = \frac{1}{\frac{1}{2}e^{-j\pi/4}} = 2e^{+j\pi/4}$

→ And again because $h[n]$ is real, there must be another zero @ $z = 2e^{-j\pi/4}$,

We have found five zeros at $z = 1, \frac{1}{2}e^{\pm j\pi/4}, 2e^{\pm j\pi/4}$

→ Since H is given to be 5th order, these are <u>all</u> of the zeros.

→ Since H is FIR, there are also 5 poles at $z = 0$.

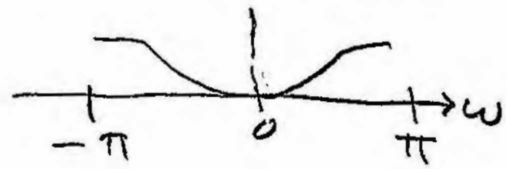


Problem 3, cont...

(b) **6/5 pts.** By visual inspection of the pole-zero plot in part (a), determine if the filter $H(z)$ is high pass, low pass, band pass, or band stop.

→ The zeros force $H(e^{j\omega})$ to be zero @ $\omega=0$ and pull $H(z)$ down in the right half of the z-plane.

→ The poles in the center pull $H(z)$ up in the left half of the z-plane... and at $\omega=\pm\pi$ in particular.



$\boxed{\Rightarrow \text{It is a High Pass Filter}}$

(c) **6/5 pts.** It is *given* that $H(z)$ has linear phase. Does $H(z)$ also have minimum phase? (*briefly justify your answer*).

No. $H(z)$ is not minimum phase because there are two zeros outside the unit circle.

4. **25/20 pts.** Use the window design method with an appropriate window function from the Table on page 10.14 of the course notes to design a causal low pass FIR digital filter that meets the following specifications:

Passband Edge Freq.	$\omega_p = 0.45\pi$ rad/sample
Stopband Edge Freq.	$\omega_s = 0.55\pi$ rad/sample
Max. Passband Ripple	$\delta_p = 0.005$
Max. Stopband Ripple	$\delta_s = 0.005$

$$\alpha_s = -20\log_{10} 0.005 = 46.0206 \text{ dB}$$

→ Rectangular and Hann can <u>not</u> meet the stopband spec.
→ Hamming and Blackman <u>can</u> meet the stopband spec.

$$\Delta\omega = \omega_s - \omega_p = 0.55\pi - 0.45\pi = 0.1\pi = \pi/10$$

Hamming: $\Delta\omega = \frac{\pi}{10} = \frac{3.32\pi}{M} \Rightarrow M = \lceil 33.2 \rceil = 34$

Blackman: $\Delta\omega = \frac{\pi}{10} = \frac{5.56\pi}{M} \Rightarrow M = \lceil 55.6 \rceil = 56$

⇒ Hamming meets the stopband spec with lower order: <u>Use Hamming</u>

$M = 34$, order $= 2M = 68$, Length $= 2M+1 = 69$.

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.45\pi + 0.55\pi}{2} = \frac{\pi}{2}$$

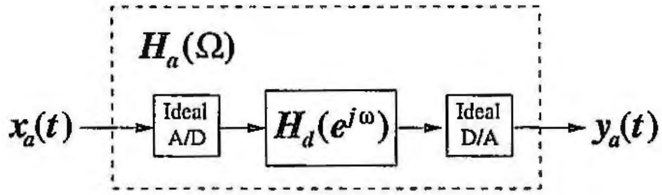
$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\sin \frac{\pi}{2} n}{\pi n}$$

$$w[n] = 0.54 + 0.46\cos\left(\frac{\pi n}{M}\right) = 0.54 + 0.46\cos\left(\frac{\pi n}{34}\right), \quad -34 \le n \le 34$$

$$h_1[n] = w[n] h_{LP}[n] = \left[0.54 + 0.46\cos\left(\frac{\pi n}{34}\right)\right] \frac{\sin\left(\frac{\pi}{2} n\right)}{\pi n}, \quad -34 \le n \le 34$$

$$\boxed{h[n] = \left\{0.54 + 0.46\cos\left[\frac{\pi(n-34)}{34}\right]\right\} \frac{\sin\left[\frac{\pi}{2}(n-34)\right]}{\pi(n-34)}, \quad 0 \le n \le 68}$$

5. **25/20 pts.** The LTI analog filter $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, an LTI digital filter $H_d(e^{j\omega})$, and an ideal D/A converter.



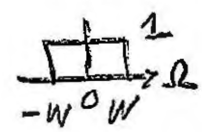
The sampling frequency of the A/D and D/A converters is $F_T = 44.1$ kHz, so $\Omega_T = 88,200\pi$ rad/sec. All input signals $x_a(t)$ are bandlimited to $|\Omega| < \Omega_T/2$ (this simply ensures that the overall structure shown in the figure will be an LTI filter).

The impulse response of the analog filter H_a is given by

$$h_a(t) = \frac{\sin(44,100\pi t)}{\pi t} - \frac{\sin(11,025\pi t)}{\pi t}.$$

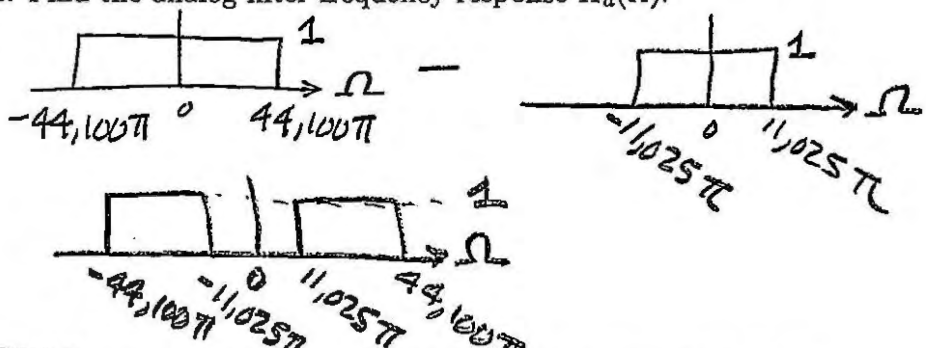
Table:

$\frac{\sin Wt}{\pi t} \Longleftrightarrow$



(a) **8/6 pts.** Find the analog filter frequency response $H_a(\Omega)$.

Table: $H_a(\Omega) =$



$$\boxed{H_a(\Omega) = \begin{cases} 1, & 11,025\pi < |\Omega| < 44,100\pi \\ 0, & \text{otherwise} \end{cases}}$$

(b) **8/7 pts.** Find the digital filter frequency response $H_d(e^{j\omega})$.

$\Omega_T = \frac{2\pi}{T} \mapsto T = \frac{2\pi}{\Omega_T}$

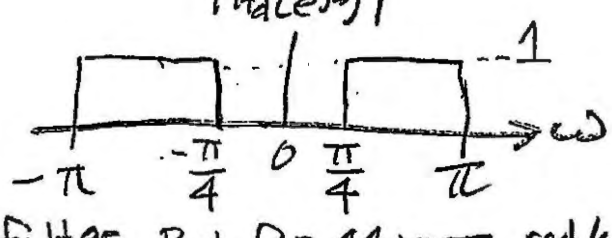
$\omega = \Omega \cdot T = 2\pi \frac{\Omega}{\Omega_T}$

$= \frac{2\pi}{88,200\pi}\Omega$

$\omega = \frac{\Omega}{44,100}$ //

$$\boxed{H_d(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} < |\omega| < \pi \\ 0, & |\omega| < \pi/4 \end{cases}}$$

$|H_d(e^{j\omega})|$



$H_d(e^{j\omega}) = H_a\left(\frac{\omega}{T}\right), |\omega| < \pi$

NOTE: $H_a(\Omega)$ looks like a bandpass filter. But $\Omega = 44,100\pi$ rad/sec gets mapped to $\omega = \pi$ rad/sample, which makes $H_d(e^{j\omega})$ High Pass.

Problem 5, cont...

(c) **9/7 pts.** Find the digital filter impulse response $h_d[n]$.

- First Way: realizing that $H(e^{j\omega})$ is 2π-periodic the graph on the previous page can be written as

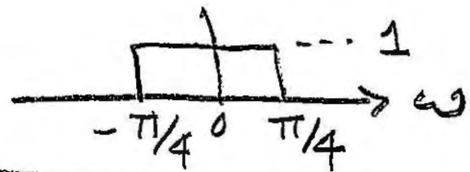
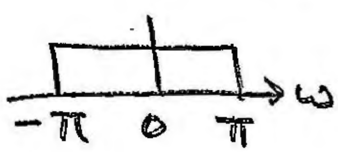
$H_d(e^{j\omega}) = 1 -$ 

Table: $\boxed{h_d[n] = \delta[n] - \dfrac{\sin(\frac{\pi}{4}n)}{\pi n}}$

- SECOND WAY: Considering $H_d(e^{j\omega})$ to be bandpass, like $H_a(\Omega)$, we may write $H_d(e^{j\omega})$ as

$H_d(e^{j\omega}) =$ 

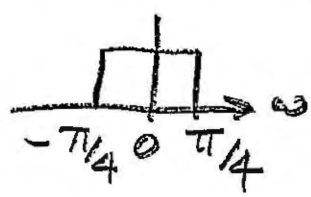
$-$ 

Table: $$h_d[n] = \frac{\sin(\pi n)}{\pi n} - \frac{\sin(\frac{\pi}{4}n)}{\pi n}$$

$$= \delta[n] - \frac{\sin(\frac{\pi}{4}n)}{\pi n} \quad ///$$

$\rightarrow$ as shown on notes pp. 3.102–3.103 (same answer as the first way)