ECE 4213/5213
Test 2

Wednesday, November 29, 2023
4:30 PM - 5:45 PM

Fall 2023 — Name: SOLUTION

Dr. Havlicek — Student Num:

Directions: This test is open book and open notes. You may also use your calculator and the course formula sheet. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ______
2. (25/20) ______
3. (25/20) ______
4. (25/20) ______
5. (25/20) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ______ Date: ______

1. **25/20 pts.** Let $x[n]$ and $h[n]$ be finite-length discrete-time signals given by

$$\begin{aligned} x[n] &= [2\ 7\ 1\ 8\ 2] \\ &= 2\delta[n] + 7\delta[n-1] + \delta[n-2] + 8\delta[n-3] + 2\delta[n-4],\ 0 \le n \le 4, \end{aligned}$$

and

$$\begin{aligned} h[n] &= [1\quad 2\ -2\ -1] \\ &= \delta[n] + 2\delta[n-1] - 2\delta[n-2] - \delta[n-3],\ 0 \le n \le 3. \end{aligned}$$

Use the DFT to find the **circular** convolution $y[n] = x[n] \circledS h[n]$.

$N=5$, so use W_5. Since $h[n]$ has a length of 4, we must pad it with one zero to extend the length to 5.

$x[n] = [2\ 7\ 1\ 8\ 2]$

$$X[k] = \sum_{n=0}^{4} x[n] W_5^{nk}$$

$$= 2 + 7W_5^k + W_5^{2k} + 8W_5^{3k} + 2W_5^{4k}$$

$h[n] = [1\ 2\ -2\ -1\ 0]$

$$H[k] = \sum_{n=0}^{4} h[n] W_5^{nk}$$

$$= 1 + 2W_5^k - 2W_5^{2k} - W_5^{3k}$$

$$Y[k] = H[k]X[k]$$

$$= (1+2W_5^k - 2W_5^{2k} - W_5^{3k})(2 + 7W_5^k + W_5^{2k} + 8W_5^{3k} + 2W_5^{4k})$$

$$\begin{aligned} = 2 &+ 7W_5^k + W_5^{2k} + 8W_5^{3k} + 2W_5^{4k} \\ &+ 4W_5^k + 14W_5^{2k} + 2W_5^{3k} + 16W_5^{4k} + 4W_5^{5k} \\ &- 4W_5^{2k} - 14W_5^{3k} - 2W_5^{4k} - 16W_5^{5k} - 4W_5^{6k} \\ &- 2W_5^{3k} - 7W_5^{4k} - W_5^{5k} - 8W_5^{6k} - 2W_5^{6k} \end{aligned}$$

$$Y[k] = 2 + 11W_5^k + 11W_5^{2k} - 6W_4^{3k} + 9W_5^{4k} - 13\underbrace{W_5^{5k}}_{1} - 12\underbrace{W_5^{6k}}_{W_5^k} - 2\underbrace{W_5^{6k}}_{W_5^{2k}}$$

$$= -11 - W_5^k + 9W_5^{2k} - 6W_5^{3k} + 9W_5^{4k}$$

$$= \sum_{n=0}^{4} y[n] W_5^{nk}$$

$$\boxed{\begin{aligned} y[n] &= [-11\ -1\ 9\ -6\ 9] \\ &= -11\delta[n] - \delta[n-1] + 9\delta[n-2] - 6\delta[n-3] + 9\delta[n-4],\ 0 \le n \le 4 \end{aligned}}$$

More Workspace for Problem 1...

"OTHER WAY":

$$Y[k] = X[k]H[k]$$

$$= (2+7W_5^k+W_5^{2k}+8W_5^{3k}+2W_5^{4k})(1+2W_5^k-2W_5^{2k}-W_5^{3k})$$

$$\begin{aligned}
= 2 &+ 4W_5^k - 4W_5^{2k} - 2W_5^{3k} \\
&+ 7W_5^k + 14W_5^{2k} - 14W_5^{3k} - 7W_5^{4k} \\
&+ W_5^{3k} + 2W_5^{3k} - 2W_5^{4k} - W_5^{5k} \\
&+ 8W_5^{3k} + 16W_5^{4k} - 16W_5^{5k} - 8W_5^{6k} \\
&+ 2W_5^{4k} + 4W_5^{5k} - 4W_5^{6k} - 2W_5^{7k}
\end{aligned}$$

$$Y[k] = 2+11W_5^k+11W_5^{2k}-6W_5^{3k}+9W_5^{4k}-13\underbrace{W_5^{5k}}_{1}-12\underbrace{W_5^{6k}}_{W_5^k}-2\underbrace{W_5^{7k}}_{W_5^{2k}}$$

$$= -11-W_5^k+9W_5^{2k}-6W_5^{3k}+9W_5^{4k} = \sum_{n=0}^{4} y[n]W_5^{nk}$$

$$\boxed{\begin{aligned} y[n] &= [-11\ \ -1\ \ 9\ \ -6\ \ 9] \\ &= -11-\delta[n-1]+9\delta[n-2]-6\delta[n-3]+9\delta[n-4],\ 0\le n\le 4\end{aligned}}$$

2. **25/20 pts.** A digital filter F has transfer function

$$F(z) = \frac{\left(1-\frac{5}{6}z^{-1}\right)\left(1-\frac{3}{2}e^{j\pi/4}z^{-1}\right)\left(1-\frac{3}{2}e^{-j\pi/4}z^{-1}\right)}{\left(1-5z^{-1}\right)\left(1-\frac{1}{2}e^{j\pi/3}z^{-1}\right)\left(1-\frac{1}{2}e^{-j\pi/3}z^{-1}\right)}, \qquad |z| > 5.$$

(a) **20/16 pts.** Find the transfer function $H(z)$ for a *new* digital filter H such that:

i. $H(z)$ has the same magnitude response as $F(z)$; i.e., $|H(e^{j\omega})| = |F(e^{j\omega})|\ \forall\ \omega \in \mathbb{R}$.

ii. $H(z)$ is causal, stable, and minimum phase.

→ For causal, stable, and minimum phase, we need all poles and zeros to be inside the unit circle.

$$F(z) = \underbrace{\frac{\left(1-5/6\, z^{-1}\right)}{\left(1-\frac{1}{2}e^{j\pi/3}z^{-1}\right)\left(1-\frac{1}{2}e^{-j\pi/3}z^{-1}\right)}}_{\text{The "good" part of } F(z)} \cdot \underbrace{\left(1-\frac{3}{2}e^{j\pi/4}z^{-1}\right)\left(1-\frac{3}{2}e^{-j\pi/4}z^{-1}\right)}_{\text{two bad zeros outside the circle}} \cdot \underbrace{\frac{1}{\left(1-5z^{-1}\right)}}_{\text{one bad pole outside the circle}}$$

$$= \underbrace{\frac{\left(1-\frac{5}{6}z^{-1}\right)}{\left(1-\frac{1}{2}e^{j\pi/3}z^{-1}\right)\left(1-\frac{1}{2}e^{-j\pi/3}z^{-1}\right)}}_{\text{Good Part}} \cdot \underbrace{\left(1-\frac{3}{2}e^{j\pi/4}z^{-1}\right)\left(1-\frac{3}{2}e^{-j\pi/4}z^{-1}\right)}_{\text{Bad Zeros}} \underbrace{\frac{\left(z^{-1}-\frac{3}{2}e^{-j\pi/4}\right)\left(z^{-1}-\frac{3}{2}e^{j\pi/4}\right)}{\left(z^{-1}-\frac{3}{2}e^{-j\pi/4}\right)\left(z^{-1}-\frac{3}{2}e^{j\pi/4}\right)}}_{\text{one}}$$

$$\cdot \underbrace{\frac{1}{\left(1-5z^{-1}\right)}}_{\text{Bad Pole}} \underbrace{\frac{\left(z^{-1}-5\right)}{\left(z^{-1}-5\right)}}_{\text{one}} \longrightarrow$$

$\overbrace{\hphantom{Problem 2, cont...}}^{\text{the good part}}$

Problem 2, cont...

$$F(z) = \frac{(1-5/6\, z^{-1})}{(1-\frac{1}{2}e^{j\pi/3}z^{-1})(1-\frac{1}{2}e^{-j\pi/3}z^{-1})}$$

$$\cdot \underbrace{(z^{-1}-\tfrac{3}{2}e^{-j\pi/4})(z^{-1}-\tfrac{3}{2}e^{j\pi/4})}_{\text{new good zeros}} \cdot \underbrace{\frac{(1-\frac{3}{2}e^{j\pi/4}z^{-1})(1-\frac{3}{2}e^{-j\pi/4}z^{-1})}{(z^{-1}-\frac{3}{2}e^{-j\pi/4})(z^{-1}-\frac{3}{2}e^{j\pi/4})}}_{\text{all pass}}$$

$$\cdot \underbrace{\frac{1}{(z^{-1}-5)}}_{\text{new good pole}} \cdot \underbrace{\frac{(z^{-1}-5)}{1-5z^{-1}}}_{\text{all pass}}$$

$$H(z) = \frac{(1-5/6\, z^{-1})(z^{-1}-\frac{3}{2}e^{-j\pi/4})(z^{-1}-\frac{3}{2}e^{j\pi/4})}{(1-\frac{1}{2}e^{j\pi/3}z^{-1})(1-\frac{1}{2}e^{-j\pi/3}z^{-1})(z^{-1}-5)}$$

$$= \frac{(1-\frac{5}{6}z^{-1})(-\frac{3}{2}e^{-j\pi/4})(1-\frac{2}{3}e^{j\pi/4}z^{-1})(-\frac{3}{2}e^{j\pi/4})(1-\frac{2}{3}e^{-j\pi/4}z^{-1})}{(1-\frac{1}{2}e^{j\pi/3}z^{-1})(1-\frac{1}{2}e^{-j\pi/3}z^{-1})(-5)(1-\frac{1}{5}z^{-1})}$$

$$H(z) = \frac{\frac{9}{4}(1-\frac{5}{6}z^{-1})(1-\frac{2}{3}e^{j\pi/4}z^{-1})(1-\frac{2}{3}e^{-j\pi/4}z^{-1})}{-5(1-\frac{1}{2}e^{j\pi/3}z^{-1})(1-\frac{1}{2}e^{-j\pi/3}z^{-1})(1-\frac{1}{5}z^{-1})} \quad //$$

(b) 5/4 pts. List the poles and zeros of your *new* transfer function $H(z)$ and specify the region of convergence.

Hint: a pole-zero plot is *not* required.

poles: $\frac{1}{2}e^{j\pi/3}$, $\frac{1}{2}e^{-j\pi/3}$, $\frac{1}{5}$

zeros: $\frac{5}{6}$, $\frac{2}{3}e^{j\pi/4}$, $\frac{2}{3}e^{-j\pi/4}$

ROC: $|z| > \frac{1}{2}$ //

3. **25/20 pts.** H is a causal, stable sixth-order Type III linear phase FIR digital filter. The impulse response $h[n]$ is real.

   $H(z)$ has a zero at $z = \frac{1}{5}e^{j2\pi/3}$.

   (a) **20/16 pts.** Give a pole-zero plot for $H(z)$.

   Hint 1: You do *not* have to find the transfer function $H(z)$.

   Hint 2: It is an FIR filter – don't forget about the poles when you make your plot!

Because H is Type III, there are zeros at $z=1$ and $z=-1$

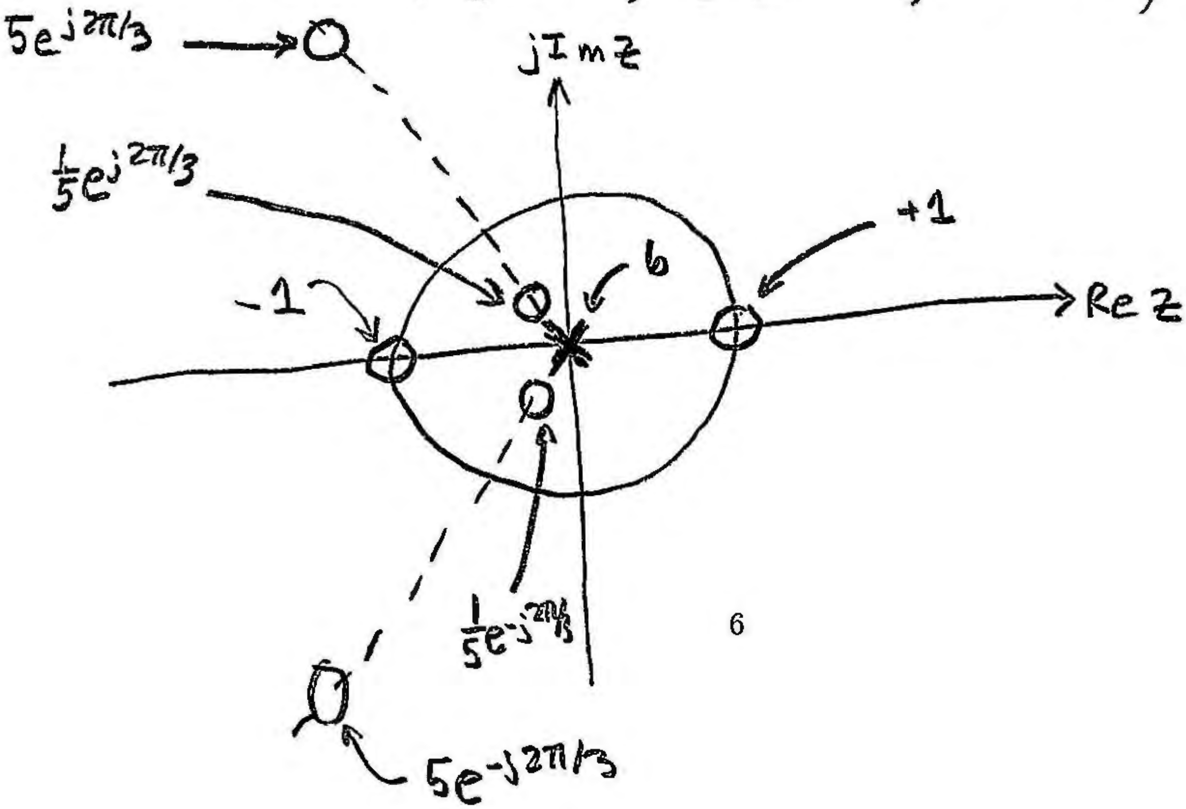
Because $h[n]$ is real and H is linear phase, the given zero at $z=\frac{1}{5}e^{j2\pi/3}$ must be accompanied by 3 additional zeros @ $z=\frac{1}{5}e^{-j2\pi/3}$, $5e^{j\pi/3}$, and $5e^{-j2\pi/3}$.

Because $N=6$, there are six zeros... so we have found them all.

Because H is FIR, there must also be six poles @ $z=0$.

poles: $z=0$ (6th order)

zeros: $z = 1, -1, \frac{1}{5}e^{j2\pi/3}, \frac{1}{5}e^{-j2\pi/3}, 5e^{j2\pi/3}, 5e^{-j2\pi/3}$



Problem 3, cont...

(b) **5/4 pts.** Determine if the filter $H(z)$ is high pass, low pass, band pass, or band stop. *Briefly justify your answer.*

Notes p. 7.51: a Type III linear phase FIR can only be bandpass.

Therefore, H is a bandpass filter

4. **25/20 pts.** Design an analog Butterworth low pass filter to meet the following analog design specification:

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{9}{10}$$

$$\sqrt{1+\varepsilon^2} = 10/9$$

$$1+\varepsilon^2 = \frac{100}{81}$$

$$\varepsilon^2 = \frac{100}{81} - 1 = \frac{100-81}{81}$$

$$\varepsilon^2 = \frac{19}{81}$$

$$\frac{1}{A} = \frac{1}{10}$$

$$A = 10$$

Passband Edge Freq.	$\Omega_p = \pi$ rad/sec
Stopband Edge Freq.	$\Omega_s = 2\pi$ rad/sec
Min. Stopband Attenuation	$1/A = 0.10$
Max. Passband Attenuation	$1/\sqrt{1+\varepsilon^2} = 0.90$

Give the analog filter transfer function $H_a(s)$.

Hint: you can leave $H_a(s)$ in factored form – i.e., you do **not** have to "multiply out" the numerator and denominator.

$$N = \left\lceil \frac{1}{2} \frac{\log_{10}\left((A^2-1)/\varepsilon^2\right)}{\log_{10}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{1}{2} \frac{\log_{10}\left[(100-1)/(19/81)\right]}{\log_{10}(2\pi/\pi)} \right\rceil$$

$$= \left\lceil \frac{1}{2} \frac{\log_{10}\left(99 \cdot \frac{81}{19}\right)}{\log_{10}(2)} \right\rceil = \left\lceil \frac{1}{2} \frac{\log_{10}(422.05\boxed{3})}{0.30\boxed{130}} \right\rceil$$

$$= \left\lceil \frac{1}{2} \frac{2.625\boxed{37}}{0.30\boxed{130}} \right\rceil = \lceil 4.36064 \rceil = 5 \; ///$$

Use (A.8b) to find Ω_c:

$$\frac{1}{1+(\Omega_s/\Omega_c)^{2N}} = \frac{1}{A^2}$$

$$1 + \frac{\Omega_s^{2N}}{\Omega_c^{2N}} = 100$$

$$\frac{\Omega_s^{10}}{\Omega_c^{10}} = 99$$

$$\Omega_c = \frac{\Omega_s}{(99)^{1/10}} = \frac{2\pi}{1.58330}$$

$$\Omega_c = 3.968\boxed{41}$$

$$p_1 = \Omega_c e^{j[\pi(5+2-1)/10]} = \Omega_c e^{j3\pi/5}$$

$$p_2 = \Omega_c e^{j[\pi(5+4-1)/10]} = \Omega_c e^{j4\pi/5}$$

$$p_3 = \Omega_c e^{j[\pi(5+6-1)/10]} = \Omega_c e^{j\pi} = -\Omega_c$$

$$p_4 = \Omega_c e^{j[\pi(5+8-1)/10]} = \Omega_c e^{j6\pi/5} = \Omega_c e^{-j4\pi/5} = p_2^*$$

$$p_5 = \Omega_c e^{j[\pi(5+10-1)/10]} = \Omega_c e^{j7\pi/5} = \Omega_c e^{-j3\pi/5} = p_1^*$$

$\longrightarrow$

More Workspace for Problem 4...

$$H_a(s) = \frac{\Omega_c^N}{\prod_{l=1}^{5} (s - p_l)}$$

$$= \frac{\Omega_c^5}{(s-\Omega_c e^{j3\pi/5})(s-\Omega_c e^{j4\pi/5})(s+\Omega_c)(s-\Omega_c e^{-j4\pi/5})(s-\Omega_c e^{-j3\pi/5})}$$

$$\boxed{H_a(s) = \frac{984.196}{(s-3.96841\, e^{j3\pi/5})(s-3.96841\, e^{j4\pi/5})(s+3.96841)(s-3.96841\, e^{-j4\pi/5}) \times (s-3.96841\, e^{-j3\pi/5})}}$$

5. **25/20 pts.** Design a causal, stable "simple" second-order IIR digital bandpass filter with a passband center frequency of $\omega_0 = \pi/3$ rad/sample and a 3 dB bandwidth of $B_W = \pi/6$.

Give the transfer function $H(z)$ and be sure to specify the ROC.

Hint: Make sure that your calculator is set for **radians** and **not** degrees!

$$\omega_0 = \arccos\beta$$

$$\beta = \cos\omega_0 = \cos\tfrac{\pi}{3} = \tfrac{1}{2} \,//$$

$$H(z) = \frac{1-\alpha}{2}\,\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

poles are at $z = re^{\pm j\phi}$,

where $r = \sqrt{\alpha}$

$= 0.759836$

$$\boxed{H(z) = \frac{0.211325(1-z^{-2})}{1-0.788675\,z^{-1}+0.577350\,z^{-2}} \qquad |z| > 0.759836}$$

$$B_W = \arccos\left(\frac{2\alpha}{1+\alpha^2}\right)$$

$$\cos B_W = \frac{2\alpha}{1+\alpha^2}$$

$$\cos B_W + \alpha^2\cos B_W = 2\alpha$$

$$[\cos B_W]\alpha^2 - 2\alpha + \cos B_W = 0$$

$a = c = \cos(B_W)$; $b = -2$

$$\alpha = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{2\pm\sqrt{4-4(\cos\frac{\pi}{6})^2}}{2\cos\pi/6}$$

$$= \frac{2\pm\sqrt{4-3}}{1.73205} = \frac{2\pm 1}{1.73205}$$

$\alpha = 1.73205$ or $\alpha = 0.577350$

For stability, we must have;

$|\alpha| < 1$

$\Rightarrow \alpha = 0.577350 \,//$

More Workspace for Problem 5...