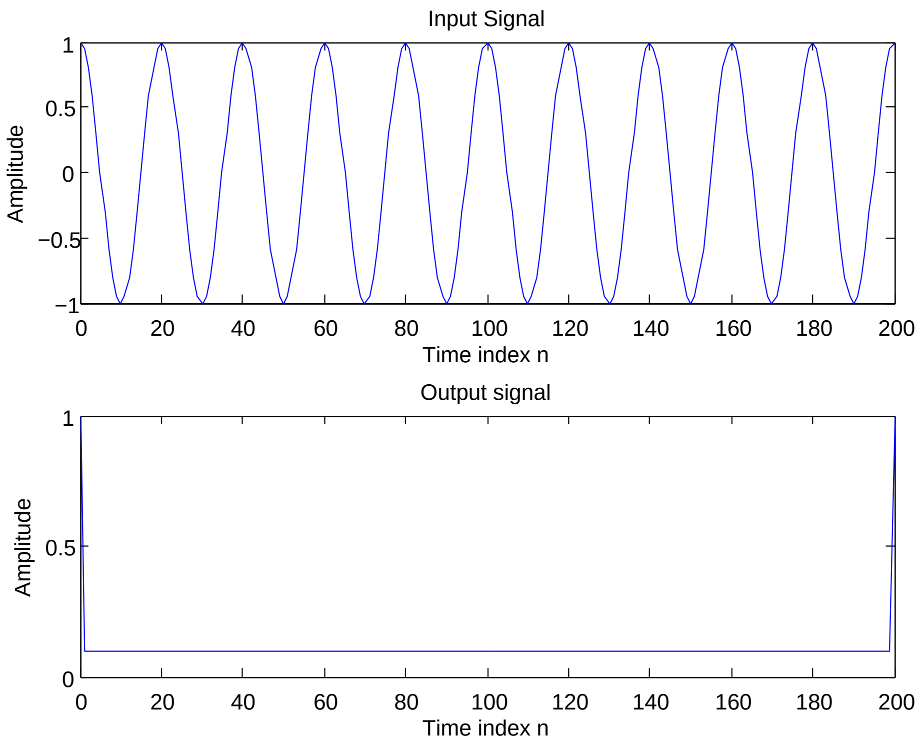
ECE 4213/5213
Homework 3 Solution

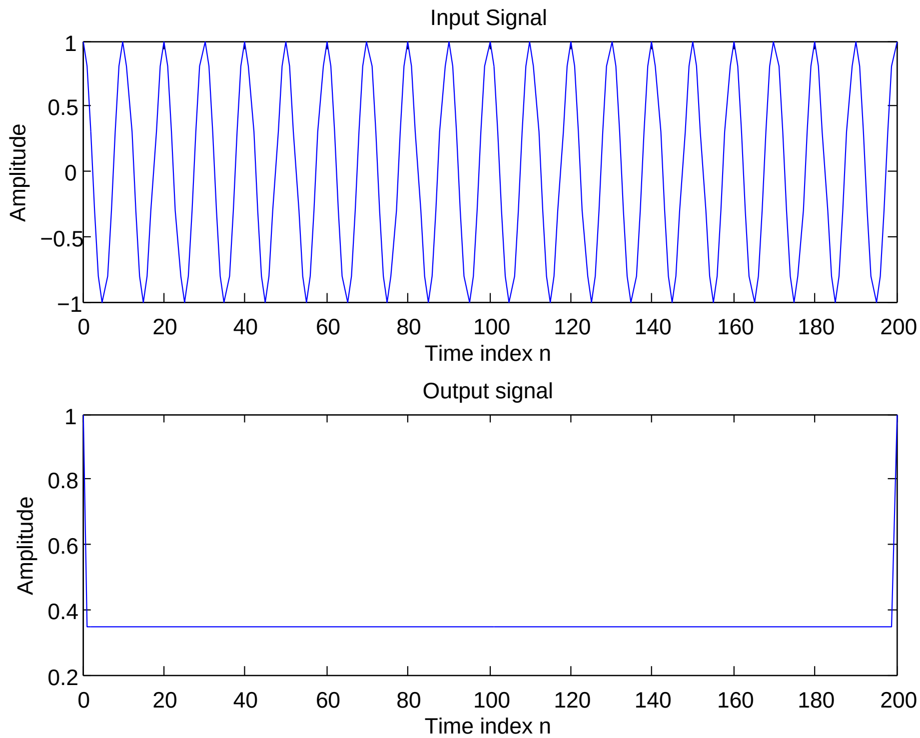
Fall 2023 Dr. Havlicek

1(a). The following graphs were obtained by running Program P2_2 with the three specified frequencies:

$f = 0.05$

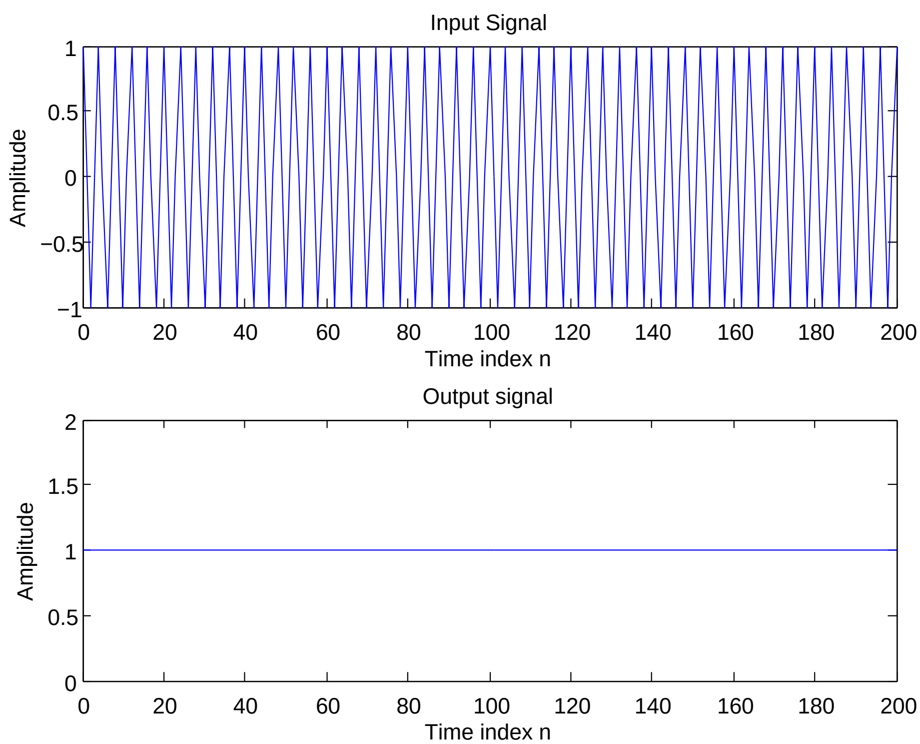


$f = 0.10$



Input Signal
Amplitude
1
0.5
0
−0.5
−1
0
20
40
60
80
100
120
140
160
180
200
Time index n
Output signal
Amplitude
1
0.8
0.6
0.4
0.2
0
20
40
60
80
100
120
140
160
180
200
Time index n

$f = 0.25$



Input Signal
Amplitude
1
0.5
0
−0.5
−1
0
20
40
60
80
100
120
140
160
180
200
Time index n
Output signal
Amplitude
2
1.5
1
0.5
0
0
20
40
60
80
100
120
140
160
180
200
Time index n

1(b). The output signals depend on the frequencies of the input signal according to: $y[n] = \sin^2(\omega) = \sin^2(2\pi f)$.

Mathematical Verification:

Let $\omega = 2\pi f$ and let the input signal be $x[n] = \cos(\omega n)$. Then (2.14) on page 21 of the course Laboratory Manual becomes

$$\begin{aligned} y[n] &= x^2[n] - x[n-1]x[n+1] \\ &= \cos^2(\omega n) - \cos[\omega(n-1)]\cos[\omega(n+1)]. \end{aligned} \tag{1}$$

We have $\cos[\omega(n-1)] = \cos(\omega n - \omega)$ and $\cos[\omega(n+1)] = \cos(\omega n + \omega)$, so

$$x[n-1]x[n+1] = \cos(\omega n - \omega)\cos(\omega n + \omega). \tag{2}$$

Applying the trigonometric identity $\cos(A+B)\cos(A-B) = \frac{1}{2}\cos(2A) + \frac{1}{2}\cos(2B)$ to (2), we obtain

$$x[n-1]x[n+1] = \frac{1}{2}\cos(2\omega n) + \frac{1}{2}\cos(2\omega). \tag{3}$$

Upon applying the trigonometric identity $\cos(2A) = 2\cos^2(A) - 1$ to the first term on the right-hand side of (3), there results

$$x[n-1]x[n+1] = \cos^2(\omega n) - \frac{1}{2} + \frac{1}{2}\cos(2\omega). \tag{4}$$

Applying the trigonometric identity $\cos(2A) = 1 - 2\sin^2(A)$ to the second term on the right-hand side of (4) then gives us

$$\begin{aligned} x[n-1]x[n+1] &= \cos^2(\omega n) - \frac{1}{2} + \frac{1}{2} - \sin^2(\omega) \\ &= \cos^2(\omega n) - \sin^2(\omega). \end{aligned} \tag{5}$$

Substituting (5) into (1), we then obtain

$$y[n] = \cos^2(\omega n) - \cos^2(\omega n) + \sin^2(\omega) = \sin^2(\omega), \tag{6}$$

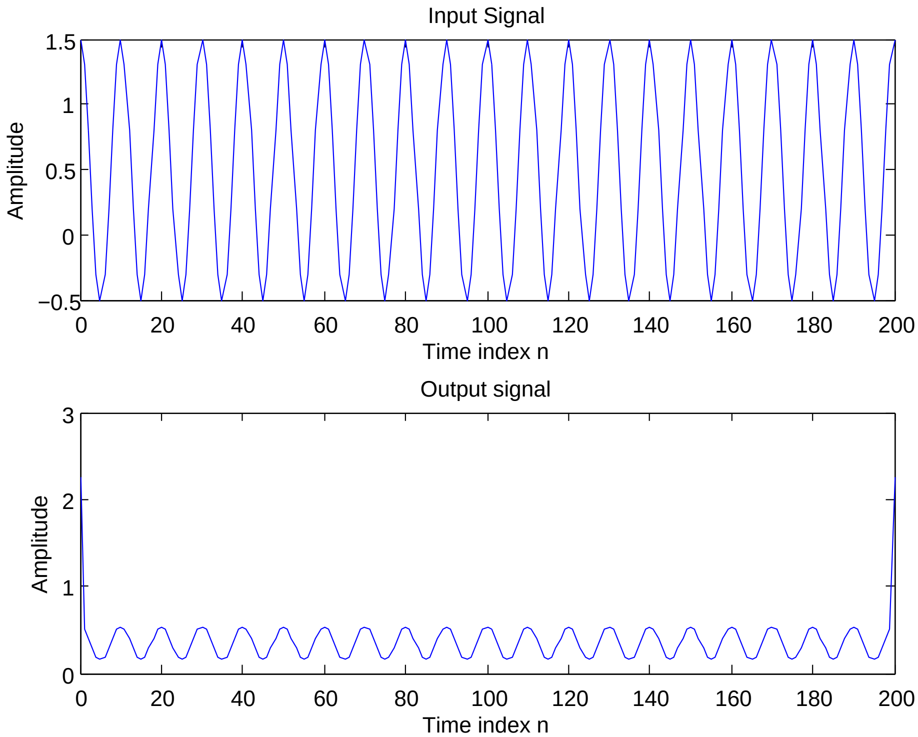
which verifies the desired result.

2(a). Listing of the modified program to make the input a cosine plus constant:

```
% Program Q2_6
%   Modified version of P2_2 to make the input a cosine plus a constant.
clf;
n = 0:200;
K = 0.5;
f = 0.1;
arg = 2*pi*f*n;
x = cos(arg) + K;
% Compute the output signal
x1 = [x 0 0];           % x1[n] = x[n+1]
x2 = [0 x 0];           % x2[n] = x[n]
x3 = [0 0 x];           % x3[n] = x[n-1]
y = x2.*x2-x1.*x3;
y = y(2:202);
% Plot the input and output signals
subplot(2,1,1)
plot(n, x)
xlabel('Time index n');ylabel('Amplitude');
title('Input Signal')
subplot(2,1,2)
plot(n,y)
xlabel('Time index n');ylabel('Amplitude');
title('Output signal');
```

Graphs obtained by running the modified program:

$$f = 0.1,\ K = 0.5$$



2(b). From Eq. (3) on page 366 of the paper by Restrepo and Quiroga, we obtained

$$y[n] = TK[\cos(\omega_0 n) + K] = \sin^2(\omega_0) + 2K[1 - \cos(\omega_0)]\cos(\omega_0 n). \tag{7}$$

I will select choice (ii) and verify the correctness of (7) by starting with Eq. (2.14) on page 21 of the course Laboratory Manual.

Let $x[n] = \cos(\omega_0 n) + K$. Eq. (2.14) of the course Laboratory Manual says that

$$y[n] = x^2[n] - x[n-1]x[n+1]. \tag{8}$$

We have that

$$x^2[n] = \cos^2(\omega_0 n) + 2K\cos(\omega_0 n) + K^2. \tag{9}$$

Working on the second term on the right-hand side of (8), we obtain

$$x[n-1] = \cos[\omega_0(n-1)] + K = \cos(\omega_0 n - \omega_0) + K$$

and

$$x[n+1] = \cos[\omega_0(n+1)] + K = \cos(\omega_0 n + \omega_0) + K,$$

so

$$x[n-1]x[n+1] = \cos(\omega_0 n - \omega_0)\cos(\omega_0 n + \omega_0) + K[\cos(\omega_0 n - \omega_0) + \cos(\omega_0 n + \omega_0)] + K^2. \quad (10)$$

Now, it was already shown in (2)-(5) above that

$$\cos(\omega_0 n - \omega_0)\cos(\omega_0 n + \omega_0) = \cos^2(\omega_0 n) - \sin^2(\omega_0). \quad (11)$$

Substituting this into (10) and applying the trigonometric identity $\cos(A+B)+\cos(A-B) = 2\cos(A)\cos(B)$ to the term in square brackets, we get

$$x[n-1]x[n+1] = \cos^2(\omega_0 n) - \sin^2(\omega_0) + 2K\cos(\omega_0 n)\cos(\omega_0) + K^2. \quad (12)$$

Substituting (9) and (12) into (8) then produces the final result

$$\begin{aligned} y[n] &= \sin^2(\omega_0) + 2K\cos(\omega_0 n) - 2K\cos(\omega_0 n)\cos(\omega_0) \\ &= \sin^2(\omega_0) + 2K[1 - \cos(\omega_0)]\cos(\omega_0 n), \end{aligned} \quad (13)$$

thereby verifying the correctness of (7).

Equations (7) and (13) provide the answer to how the output signal $y[n]$ depends on the DC value K. Compared to the case where $x[n] = \cos(\omega_0 n)$, there is an extra term added to $y[n]$. This extra term is a cosine having the same frequency as the input and having an amplitude of $2K[1 - \cos(\omega_0)]$. This term adds a ripple to $y[n]$ which may be plainly seen in the graph on the preceding page. The amplitude of the ripple is proportional to K.

3(a). I selected the paper:

P. Maragos, J.F. Kaiser, and T.F. Quatieri, "On separating amplitude from frequency modulations using energy operators," in *Proc. IEEE Int'l. Conf. Acoust., Speech, Signal Proc.*, San Francisco, CA, Mar. 23-26, 1992, vol. II, pp. 1-4.

Another good one is:

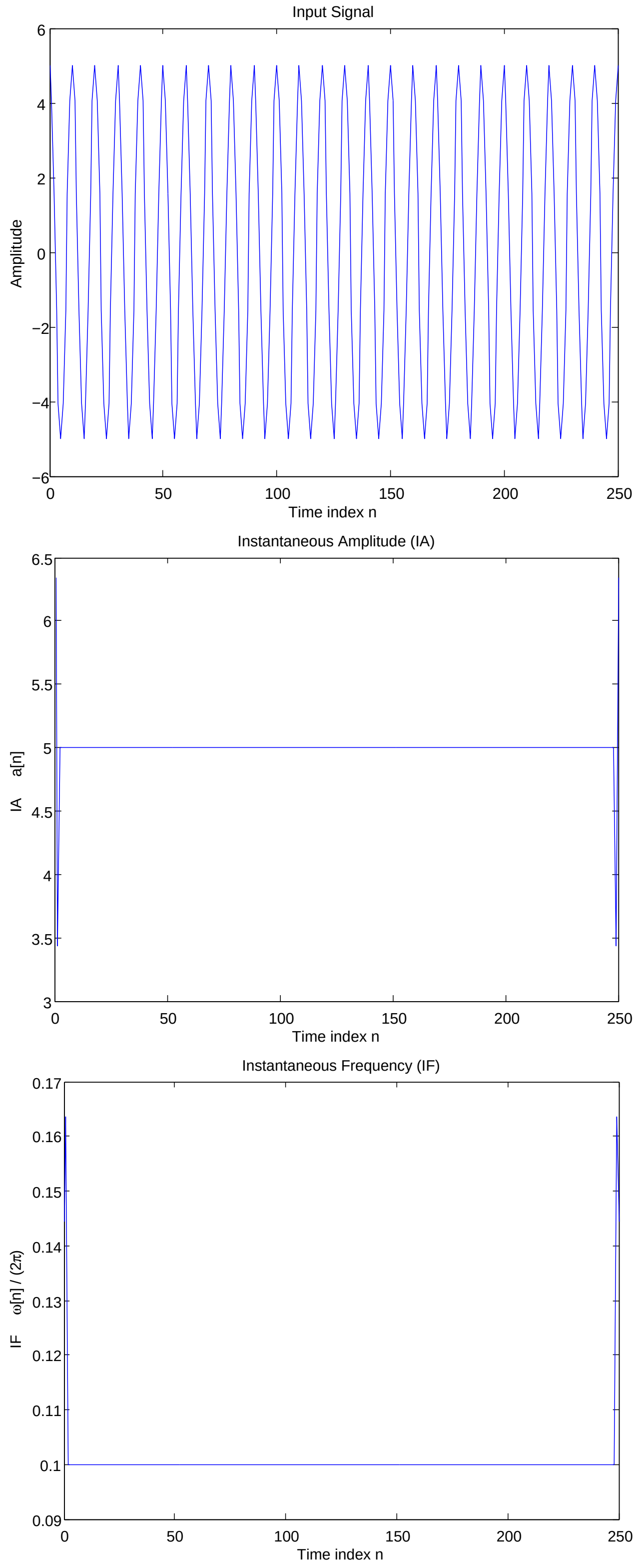
P. Maragos, J.F. Kaiser, and T.F. Quatieri, "Energy separation in signal modulations with applications to speech analysis," *IEEE Trans. Signal Process.*, vol. 41, no. 10, Oct. 1993, pp. 3024-3051.

3(b). Listing of Matlab program to implement DESA-1:

```
% Program DESA1
%   Implement the DESA-1 discrete energy separation algorithm
clear;
clf;
% Generate the input signal
n = 0:250;
A = 5.0;
f = 0.1;
w = 2*pi*f;
x = A*cos(w*n);

x2 = [0 x 0];                   % zero pad x[n]
y = [0 x 0] - [0 0 x];          % y[n] = x[n] - x[n-1]
z = [x 0 0] - [0 x 0];          % z[n] = x[n+1] - x[n]

% TK operator on y[n]
ynp1 = [y 0 0];       % y[n+1]
yn   = [0 y 0];       % y[n]
ynm1 = [0 0 y];       % y[n-1]
Psi_y = yn.*yn-ynm1.*ynp1;

% TK operator on z[n]
znp1 = [z 0 0];       % z[n+1]
zn   = [0 z 0];       % z[n]
znm1 = [0 0 z];       % z[n-1]
Psi_z = zn.*zn-znm1.*znp1;

% TK operator on x[n]
x2np1 = [x2 0 0];       % x2[n+1]
```

```
x2n   = [0 x2 0];       % x2[n]
x2nm1 = [0 0 x2];       % x2[n-1]
Psi_x2 = x2n.*x2n-x2nm1.*x2np1;

% Resize TK outputs to original size of x[n]
Psi_y = Psi_y(3:length(Psi_y)-2);
Psi_z = Psi_z(3:length(Psi_z)-2);
Psi_x2 = Psi_x2(3:length(Psi_x2)-2);

% Estimate the IA and IF
IFhat = acos(1 - (Psi_y + Psi_z)./(4*Psi_x2))/(2*pi);
IAhat = sqrt(Psi_x2./(1 - (1 - (Psi_y + Psi_z)./(4*Psi_x2)).^2));

% Plot the input and output signals
figure(1);
plot(n,x);
axis([0 250 -6 6]);
xlabel('Time index n');ylabel('Amplitude');
title('Input Signal');

figure(2);
plot(n,IAhat);
xlabel('Time index n');ylabel('IA     a[n]');
title('Instantaneous Amplitude (IA)');

figure(3);
plot(n,IFhat);
xlabel('Time index n');ylabel('IF     \omega[n] / (2\pi)');
title('Instantaneous Frequency (IF)');
```

Plots obtained by running the program with the input signal $x[n] = 5\cos[2\pi(0.1)n]$:



3(c). Plots obtained by running the program on an AM-FM chirp signal: