ECE 4213/5213

DSP

HW 5 Solution

HAVLICEK
3.4) \[ h(t) = (2\pi\sigma^2)^{-1/2} \exp \left[ -\left( t - \mu \right)^2 / 2\sigma^2 \right] \]

\[ H(\Omega) = \int\sum_{t} h(t)^2 dt = \int_{-\infty}^{\infty} h(t) e^{-j\Omega t} dt \]

\[ = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} e^{-\left( t - \mu \right)^2 / 2\sigma^2} e^{-j\Omega t} dt \]

\[ = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} \exp \left[ -\left( (t^2 + 2t\mu + \mu^2) / 2\sigma^2 - j\Omega t \right) \right] dt \]

\[ = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2\sigma^2} t^2 + \frac{\mu}{\sigma^2} t - j\Omega t - \frac{\mu^2}{2\sigma^2} \right] dt \]

\[ = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2\sigma^2} t^2 + \frac{j\Omega - \mu}{\sigma^2} t - \frac{\mu^2}{2\sigma^2} \right] dt \quad (*) \]

In my Schaum's Math Handbook, formula (18.75) says:

\[ \int_{-\infty}^{\infty} e^{-ax^2+bx+c} \, dx = \sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/4a} \]

Comparing to (*), we have:

\[ a = \frac{1}{2\sigma^2} \quad b = j\Omega - \frac{\mu}{\sigma^2} \quad c = \frac{\mu^2}{2\sigma^2} \]

So \((*)\) = \[ (2\pi\sigma^2)^{-1/2} (\pi)^{1/2} (2\sigma^2)^{1/2} \exp \left\{ \left[ j\Omega - \frac{\mu}{\sigma^2} \right]^2 - 4\left( \frac{1}{2\sigma^2} \right) \left( \frac{\mu^2}{2\sigma^2} \right) \right\} \]

\[ \times \frac{1}{4} (2\sigma^2)^{1/2} \right\} \]

\[ = \exp \left\{ \left( j\Omega - \frac{\mu}{\sigma^2} \right)^2 \left( \frac{1}{2\sigma^2} \right) - \frac{\mu^2}{2\sigma^2} \right\} \]

\[ \rightarrow \]
\[ \exp \left[ -\alpha^2 - 2j\alpha \frac{\mu^2}{\sigma^2} + \frac{\mu^2}{2\sigma^2} \right] \]

\[ = \exp \left[ -\frac{\alpha^2 \sigma^2}{2} - j\alpha \mu + \frac{\mu^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} \right] \]

\[ = e^{-\alpha^2 \sigma^2 / 2} e^{-j\alpha \mu} \]

Gaussian in \( \alpha \)  

linear phase term resulting from the time shift \((t-\mu)^2\) in \( h(t) \).

\[ \Rightarrow \text{So the transform of a Gaussian is Gaussian.} \]

If \( \mu = 0 \), then

\[ h(\alpha) = e^{-\alpha^2 \sigma^2 / 2} \]
3.6a)

Thus: if $X_a(t) \Longleftrightarrow X_a(o)$, then, for $t_0 \in \mathbb{R}$,
$$X_a(t-t_0) \Longleftrightarrow X_a(o)e^{-j2\pi t_0}.$$ 

**Proof:** Let $X_a(t) \Longleftrightarrow X_a(o)$. Then
$$\mathcal{F}\{X_a(t-t_0)\} = \int_{-\infty}^{\infty} X_a(t-t_0)e^{-j\omega t} dt$$

Let $u = t-t_0$. Then $du = dt$, $t = u + t_0$, and $dt = du$. When $t \to \infty$, $u \to \infty$. When $t \to -\infty$, $u \to -\infty$. So
$$\mathcal{F}\{X_a(t-t_0)\} = \int_{-\infty}^{\infty} X_a(u)e^{-j\omega(u+t_0)} du$$
$$= \int_{-\infty}^{\infty} X_a(u)e^{-j\omega u} e^{-j\omega t_0} du$$
$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} X_a(u)e^{-j\omega u} du$$
$$= e^{-j\omega t_0} X_a(o).$$

QED
3.6e) Theorem: if \( x_a(t) \xrightarrow{\mathcal{F}} X_a(\omega) \), then 
\[
\frac{d}{dt} x_a(t) \xrightarrow{\mathcal{F}} j \omega X_a(\omega).
\]

Proof: Let \( x_a(t) \xrightarrow{\mathcal{F}} X_a(\omega) \). Then 
\[
x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\omega) e^{j\omega t} d\omega.
\]
\[
\frac{d}{dt} x_a(t) = \frac{d}{dt} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\omega) e^{j\omega t} d\omega \right\}
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\omega) \{ \frac{d}{dt} e^{j\omega t} \} d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\omega) (j\omega) e^{j\omega t} d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ j\omega X_a(\omega) \} e^{j\omega t} d\omega
\]
\[
= \mathcal{F}^{-1} \{ j\omega X_a(\omega) \}
\]

\( \mathcal{Q.E.D.} \)
3.13) \( y[n] = \alpha^{n+1}, \quad |\alpha| < 1. \)

\[
Y(e^{i\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-i\omega n} = \sum_{n=-\infty}^{\infty} \alpha^{n+1} e^{-i\omega n}
\]

\[
= \sum_{n=-\infty}^{\infty} \alpha^{-n} e^{-i\omega n} + \sum_{n=0}^{\infty} \alpha^{n} e^{-i\omega n}
\]

\[
= \sum_{k=-\infty}^{\infty} (\alpha e^{i\omega})^{-k} + \sum_{n=0}^{\infty} (\alpha e^{i\omega})^{n}
\]

\[
= \lim_{A \to \infty} \left[ \frac{\alpha e^{i\omega} - (\alpha e^{i\omega})^{A+1}}{1 - \alpha e^{i\omega}} \right] + \lim_{A \to \infty} \left[ \frac{1 - (\alpha e^{-i\omega})^{A+1}}{1 - \alpha e^{-i\omega}} \right]
\]

\[
\Rightarrow \text{since } |\alpha| < 1 \text{ and } |\alpha e^{i\omega}| = |e^{-i\omega}| = 1,
\]

\[
\lim_{A \to \infty} (\alpha e^{i\omega})^{A+1} = \lim_{A \to \infty} (\alpha e^{-i\omega})^{A+1} = 0.
\]

\[
= \frac{\alpha e^{i\omega}}{1 - \alpha e^{i\omega}} + \frac{1}{1 - \alpha e^{-i\omega}}
\]

\[
= \frac{\alpha e^{i\omega}}{1 - \alpha e^{i\omega}} \frac{1 - \alpha e^{-i\omega}}{1 - \alpha e^{-i\omega}} + \frac{1}{1 - \alpha e^{-i\omega}} \frac{1 - \alpha e^{i\omega}}{1 - \alpha e^{i\omega}}
\]

\[
= \frac{\alpha e^{i\omega} - \alpha^{2} + 1 - \alpha e^{i\omega}}{1 - \alpha e^{i\omega} - \alpha e^{-i\omega} + \alpha^{2}} = \frac{1 - \alpha^{2}}{1 - \alpha(e^{i\omega} + e^{-i\omega}) + \alpha^{2}}
\]

\[
= \frac{1 - \alpha^{2}}{1 - 2\alpha \cos \omega + \alpha^{2}}
\]
3. (14 a) Let $x[n]$ be real and even.

Then $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$= \sum_{n=-\infty}^{\infty} x[n] \{ \cos \omega n - j \sin \omega n \}$

$= \sum_{n=-\infty}^{\infty} x[n] \cos \omega n - j \sum_{n=-\infty}^{\infty} x[n] \sin \omega n$

$= \sum_{n=-\infty}^{\infty} x[n] \cos \omega n$

This term is zero because $x[n] \sin \omega n$ is the product of an even function ($x[n]$) and an odd function ($\sin \omega n$). So overall, it's odd and it sums to zero.

$\Rightarrow$ This shows that $X(e^{j\omega})$ is real and even.

So $x[n] = \text{IDFT} \{ X(e^{j\omega}) \} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \{ \cos \omega n + j \sin \omega n \} d\omega$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cos \omega n d\omega + \frac{j}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \sin \omega n d\omega$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cos \omega n d\omega$

$\text{even}$

$= \frac{1}{\pi} \int_{0}^{\pi} X(e^{j\omega}) \cos \omega n d\omega$ //
3.14b) Let \( X[n] \) be real and odd. Then
\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} X[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} X[n] (\cos \omega n - j\sin \omega n)
\]

\[
= \sum_{n=-\infty}^{\infty} X[n] \cos \omega n - j \sum_{n=-\infty}^{\infty} X[n] \sin \omega n
\]

odd: \( \Sigma = 0 \).

\[
= -j \sum_{n=-\infty}^{\infty} X[n] \sin \omega n
\]

This is pure imaginary and odd in \( \omega \).

\( \Rightarrow \) \( X(e^{j\omega}) \) is pure imaginary and odd. So
\[
x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \{\cos \omega n + j\sin \omega n\} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cos \omega n d\omega + \frac{j}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \sin \omega n d\omega
\]

odd integrand.

\( S = 0 \).

\[
= \frac{j}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \sin \omega n d\omega = \frac{j}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \sin \omega n d\omega
\]

odd x odd = even

\[
= \frac{j}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \sin \omega n d\omega
\]
3.15 \( x[n] = A \alpha^n \cos(\omega_0 n + \phi) u[n] \), \( A, \alpha, \omega_0, \phi \in \mathbb{R}, |\alpha| < 1 \).

\[
= A \alpha^n \frac{e^{j(\omega_0 n + \phi)} + e^{-j(\omega_0 n + \phi)}}{2} u[n]
\]

\[
= \frac{A}{2} \alpha^n e^{j\omega_0 n} u[n] + \frac{A}{2} \alpha^n e^{-j\omega_0 n} u[n]
\]

\[
= \frac{Ae^{j\phi}}{2} \alpha^n e^{j\omega_0 n} u[n] + \frac{Ae^{-j\phi}}{2} \alpha^n e^{-j\omega_0 n} u[n] \quad (*)
\]

Table: \( \alpha^n u[n] \leftrightarrow \frac{1}{1 - \alpha e^{-j\omega}} \)

Freq. shift property: \( e^{j\omega_0 n} \alpha^n u[n] \leftrightarrow \frac{1}{1 - \alpha e^{-j(\omega - \omega_0)}} \)

\( e^{-j\omega_0 n} \alpha^n u[n] \leftrightarrow \frac{1}{1 - \alpha e^{-j(\omega + \omega_0)}} \)

Plug the last two results into (*):

\[
X(e^{j\omega}) = \frac{Ae^{j\phi}}{2} \frac{1}{1 - \alpha e^{-j(\omega - \omega_0)}} + \frac{Ae^{-j\phi}}{2} \frac{1}{1 - \alpha e^{-j(\omega + \omega_0)}}
\]
3.16d) \( x_4(r) = n x^n u(n+2) \), \( |\lambda| < 1 \).

\[
\begin{align*}
X_4(e^{i\omega}) &= \sum_{n=-\infty}^{\infty} x_4(n) e^{-i\omega n} = \sum_{n=-2}^{\infty} n x^n e^{-i\omega n} \\
&= -2x^{-2} e^{i2\omega} - x^{-1} e^{i\omega} + \sum_{n=-1}^{\infty} n x^n e^{-i\omega n} \\
&= -2x^{-2} e^{i2\omega} - x^{-1} e^{i\omega} + \sum_{n=0}^{\infty} n (xe^{-i\omega})^n \\
&= -2x^{-2} e^{i2\omega} - x^{-1} e^{i\omega} + \frac{xe^{-i\omega}}{(1-xe^{-i\omega})^2}
\end{align*}
\]
3.16e) \( \chi_{5}(n) = \alpha^n u(-n-1), \quad |\alpha| > 1 \).

\[
\begin{align*}
X_5(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \chi_5(n) e^{-j\omega n} = \sum_{n=-\infty}^{-1} \alpha^n e^{-j\omega n} \\
&= \sum_{n=-\infty}^{-1} (\alpha e^{-j\omega})^n \\
&= \sum_{k=1}^{\infty} (\alpha e^{-j\omega})^{-k} = \sum_{k=1}^{\infty} \left(\frac{e^{j\omega}}{\alpha}\right)^k \\
&= \lim_{A \to \infty} \frac{\frac{e^{j\omega}}{\alpha} - \left(\frac{e^{j\omega}}{\alpha}\right)^{A+1}}{1 - \frac{e^{j\omega}}{\alpha}} \\
&= \frac{e^{j\omega}/\alpha}{1 - e^{j\omega}/\alpha}, \quad \frac{1}{\alpha} = \frac{e^{j\omega}}{\alpha - e^{j\omega}}
\end{align*}
\]
Theorem: If \( x_1[n] \leftrightarrow X_1(e^{j\omega}) \) and \( x_2[n] \leftrightarrow X_2(e^{j\omega}) \) and if \( c_1, c_2 \in \mathbb{C} \) are constants, then
\[
x_3[n] = c_1 x_1[n] + c_2 x_2[n] \leftrightarrow c_1 X_1(e^{j\omega}) + c_2 X_2(e^{j\omega})
\]

Proof: \[
x_3(e^{j\omega}) = \mathcal{F}\{x_3[n]\}
= \sum_{n=-\infty}^{\infty} x_3[n] e^{-j\omega n}
= \sum_{n=-\infty}^{\infty} (c_1 x_1[n] + c_2 x_2[n]) e^{-j\omega n}
= \sum_{n=-\infty}^{\infty} c_1 x_1[n] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} c_2 x_2[n] e^{-j\omega n}
= c_1 \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\omega n} + c_2 \sum_{n=-\infty}^{\infty} x_2[n] e^{-j\omega n}
= c_1 X_1(e^{j\omega}) + c_2 X_2(e^{j\omega})
\]

QED.
Theorem: if \( x[n] \xrightarrow{DTFT} X(e^{j\omega}) \),

then \( e^{j\omega_0}x[n] \xrightarrow{DTFT} X(e^{j(\omega-\omega_0)}) \)

Proof: 
\[
\mathcal{F}\{e^{j\omega_0}x[n]\} = \sum_{n=-\infty}^{\infty} e^{j\omega_0} x[n] e^{-j\omega n}
\]
\[
= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\omega_0)n}
\]
\[
= \left[ \sum_{n=-\infty}^{\infty} x[n] e^{-j\psi n} \right]_{\psi = \omega - \omega_0}
\]
\[
= X(e^{j\psi}) \bigg|_{\psi = \omega - \omega_0}
\]
\[
= X(e^{j(\omega-\omega_0)})
\]

QED
The fundamental period of $X(e^{j\omega})$ is given in Fig. P3.3 on p. 138 of the text:

$$X(e^{j\omega})$$

The signal we are interested in is $x_2[n] = x[n]e^{-j\frac{\pi}{5}n}$. By the frequency shift property of the DTFT with $\omega_0 = -\frac{\pi}{5}$, the DTFT of $x_2[n]$ is given by

$$x_2(e^{j\omega}) = X(e^{j(\omega + \frac{\pi}{5})})$$

The effect is to shift the graph of $X(e^{j\omega})$ to the right by $\omega_0 = -\frac{\pi}{5}$:

$$\begin{align*}
-\frac{2\pi}{3} - \frac{\pi}{5} &= -\frac{10\pi}{15} - \frac{3\pi}{15} = \frac{-13\pi}{15} \\
-\frac{7\pi}{3} - \frac{\pi}{5} &= -\frac{35\pi}{15} - \frac{3\pi}{15} = \frac{-38\pi}{15} \\
0 - \frac{\pi}{5} &= -\frac{\pi}{5} = -\frac{2\pi}{10} \\
\frac{\pi}{2} - \frac{\pi}{5} &= \frac{5\pi}{10} - \frac{2\pi}{10} = \frac{3\pi}{10}
\end{align*}$$
Let $H$ be a discrete-time LTI system with impulse response $h[n]$:

$$x[n] \rightarrow \boxed{\text{LTI}}_H \rightarrow y[n] = x[n] * h[n].$$

Let the input be $x[n] = u(n) = z^n$ for an arbitrary but fixed constant $z \in \mathbb{C}$. Then the output is given by

$$y[n] = H\{x[n]\}^2 = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = \sum_{k=-\infty}^{\infty} h[k] z^{-k} z^n$$

$$= z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

The input depends on our particular number $z$ and on the impulse response $h[k].$

$\Rightarrow$ This shows that, for any fixed number $z$, the output is given by the input $x[n]$ times a complex number. Thus, the input $x[n] = z^n$ is an Eigen-function of any LTI discrete-time system.

Note: The eigenvalue is precisely $H(z).$
Now we consider the input $u[n] = z^n u[n]$ for some arbitrary but fixed number $z \in \mathbb{C}$. Here, $u[n]$ is the unit step function.

The output in this case is

$$y[n] = u[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] u[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} u[n-k]$$

$$= \sum_{k=-\infty}^{n} h[k] z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{n} h[k] z^{-k}$$

Although the term $A$ has the same form as the input, the term $B$ is not a constant... it depends on $n$, so it is in general a different number for each $n$.

$\Rightarrow$ Therefore, $u[n]$ is not an eigenfunction in general.
For $H(e^{j\omega})$ to have zero phase, the imaginary part must be zero $\forall \omega \in \mathbb{R}$.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jn\omega}$$

$$= a_1 e^{-j3\omega} + a_2 e^{-j2\omega} + a_3 e^{-j\omega} + a_4 + a_5 e^{j\omega} + a_6 e^{j2\omega}$$

$$= a_1 \cos 3\omega - ja_1 \sin 3\omega + a_2 \cos 2\omega - ja_2 \sin 2\omega + a_3 \cos \omega - ja_3 \sin \omega + a_4$$

$$+ a_5 \cos \omega + ja_5 \sin \omega + a_6 \cos 2\omega + ja_6 \sin 2\omega$$

$$= \left[ a_1 \cos 3\omega + (a_2 + a_6) \cos 2\omega + (a_3 + a_5) \cos \omega + a_4 \right]$$

$$+ j \left[ -a_1 \sin 3\omega + (a_6 - a_2) \sin 2\omega + (a_5 - a_3) \sin \omega \right]$$

To have $\text{Im} \left\{ H(e^{j\omega}) \right\} = 0 \ \forall \omega \in \mathbb{R}$, we need

$$-a_1 \sin 3\omega + (a_6 - a_2) \sin 2\omega + (a_5 - a_3) \sin \omega = 0 \ \forall \omega$$

$ightarrow$ since the sines are orthogonal, each term must be zero.

$$\Rightarrow a_1 = 0 \ ; \ a_2 = a_6 \ ; \ a_3 = a_5 \ .$$

Notes: this means that $\text{Re} \left\{ H(e^{j\omega}) \right\} = 2a_2 \cos 2\omega + 2a_3 \cos \omega + a_4$.

Since this is a signed function that is $< 0$ for some $\omega$, $H(e^{j\omega})$ actually has generalized zero phase. For true zero phase, additional constraints must be placed on $a_4$ to ensure that $\text{Re} \left\{ H(e^{j\omega}) \right\} > 0 \ \forall \omega$. 
4.62) \[ h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] + a_4 \delta[n-3] + a_5 \delta[n-4] + a_6 \delta[n-5] + a_7 \delta[n-6] + a_8 \delta[n-7]. \]

\[ H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jn\omega} = a_1 + a_2 e^{-j\omega} + a_3 e^{-j2\omega} + a_4 e^{-j3\omega} + a_5 e^{-j4\omega} + a_6 e^{-j5\omega} + a_7 e^{-j6\omega} + a_8 e^{-j7\omega}. \]

- Factor out a linear phase term \( e^{-j7\omega/2} \):

\[ H(e^{j\omega}) = e^{-j7\omega/2} \left[ a_1 e^{j7\omega/2} + a_2 e^{j5\omega/2} + a_3 e^{j3\omega/2} + a_4 e^{j\omega/2} + a_5 e^{-j\omega/2} + a_6 e^{-j3\omega/2} + a_7 e^{-j5\omega/2} + a_8 e^{-j7\omega/2} \right] \]

- The conjugate exponentials can be combined into real-valued cosines if \( a_1 = a_8, \ a_2 = a_7, \ a_3 = a_6, \ and \ a_4 = a_5 \), giving

\[ H(e^{j\omega}) = \left[ 2a_1 \cos \frac{7}{2} \omega + 2a_2 \cos \frac{5}{2} \omega + 2a_3 \cos \frac{3}{2} \omega + 2a_4 \cos \frac{1}{2} \omega \right] e^{-j\frac{7}{2} \omega}, \] which has a generalized linear phase as required.

So, \( H(e^{j\omega}) \) has generalized linear phase provided that:

\[ a_1 = a_8, \ a_2 = a_7, \ a_3 = a_6, \ and \ a_4 = a_5. \]
Consider a discrete-time "factor of $L" \]
upsample where $L \in \mathbb{N}$:

$$x[n] \rightarrow \uparrow L \rightarrow x_L[n]$$

The input-output relation for this system is given in (2.23) on page 54 of the book:

$$x_L[n] = \begin{cases} x[n/L], & n = kL, \ k \in \mathbb{Z} \\ 0, & \text{other} \end{cases}$$

Let $X(e^{j\omega})$ be the DTFT of $X[n]$, so that

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}.$$ 

Then $X_L(e^{j\omega}) = \mathcal{F}\{x_L[n]\}$

$$= \sum_{n=-\infty}^{\infty} x_L[n] e^{-j\omega n} \quad (\ast)$$

To see how this works, consider the case $L = 3$, we have

$$X_L(e^{j\omega}) = \ldots + x[-1]e^{j3\omega} + Oe^{j2\omega} + Oe^{j\omega}$$

$$+ x[0]e^{j0} + Oe^{j\omega} + Oe^{j2\omega}$$

$$+ \underbrace{x[1]e^{j3\omega}}_{x_L[3]} + Oe^{j4\omega} + Oe^{j5\omega} + \ldots$$

$$+ \underbrace{x[2]e^{j6\omega}}_{x_L[6]} + Oe^{j7\omega} + \ldots$$

$$+ \underbrace{x[3]e^{j9\omega}}_{x_L[9]} + Oe^{j10\omega} + \ldots$$
For a general $L \leq N$, the samples $x_n[n]$ can be arranged in groups of $L$. In each group, there will be $L-1$ zero samples and one sample that is equal to one of the samples $X[n]$.

Therefore, the terms in (*) can also be divided into groups of $L$. In each group, there will be only one nonzero term:

\[
X_n(e^{j\omega}) = (*) = \sum_{n=-\infty}^{\infty} x_n[n] e^{-j\omega n} \quad \text{Let } m = \frac{n}{L}
\]

\[
= \sum_{m=-\infty}^{\infty} (x_{mL}[mL] e^{-j\omega mL} + x_{mL+1}[mL+1] e^{-j\omega (mL+1)} + x_{mL+2}[mL+2] e^{-j\omega (mL+2)} + \ldots + x_{mL+(m+1)L-1}[mL+(m+1)L-1] e^{-j\omega [m(m+1)L-1]})
\]

\[
= \sum_{m=-\infty}^{\infty} \left( x_m[e^{-j\omega mL}] + o e^{-j\omega (mL+1)} + o e^{-j\omega (mL+2)} + \ldots + o e^{-j\omega [m(m+1)L-1]} \right)
\]

\[
= \sum_{m=-\infty}^{\infty} x_m[e^{-j\omega mL}]
\]

\[
= \sum_{m=-\infty}^{\infty} x_m[e^{-j(\omega L)m}]
\]

So \(X_n(e^{j\omega}) = X(e^{j\omega L})\)