ECE 4213/5213

DSP

HW 6 SOLUTION

HAVLICEK

3.1)

d) $x[n] = \delta[n]$

$$\boxed{\text{Table: } X(z) = 1, \quad \text{all } z.}$$

e) $x[n] = \delta[n-1]$

Time shift property: $X(z) = z^{-1} \cdot 1 = z^{-1} = \frac{1}{z}$

- Compared to part (d), the time shift here introduces a pole at $z=0$. So the point $z=0$ can no longer be included in the ROC.

- Here's a more general way to think about that: for any $x[n]$, the z-transform is

$$X(z) = \sum_{n \in \mathbb{Z}} x[n] z^{-n}$$
$$= \ldots x[-2] z^{2} + x[-1] z^{1} + x[0] z^{0} + x[1] z^{-1} + x[2] z^{-2} \ldots$$

→ For negative n's, $x[n]$ is times <u>positive</u> powers of z.

→ If $x[n]$ is nonzero for any negative n's, then the point $z = \infty$ <u>can't</u> be in the ROC of $X(z)$.

→ For positive n's, $x[n]$ is times negative powers of z... i.e., $x[n]$ is times powers of $\frac{1}{z}$.

→ If $x[n]$ is nonzero for any negative values of n, then the point $z=0$ can't be in the ROC of $X(z)$.

- Getting back to the problem, we know that $\delta[n] \overset{z}{\longleftrightarrow} 1$, all z. For $\delta[n-1]$, the time shift property tells us that the transform changes to z^{-1}. And since $\delta[n-1]$ is nonzero for a positive value of n ($n=1$), the point $z=0$ can no longer be in the ROC.

- Thus,

$$\boxed{X(z) = z^{-1}, \quad |z| > 0}$$

3.1 (f). Similar to part (e), we have immediately from the time shift property that

$$\boxed{X(z) = z, \quad |z| < \infty}$$

3.3 a) $x_a[n] = \alpha^{|n|}$, $\alpha \in \mathbb{C}$, $0 < |\alpha| < 1$ $\boxed{3.3a)-1}$

Method I: from the definition of $X(z)$:

$$X_a(z) = \sum_{n=-\infty}^{\infty} x_a[n] z^{-n} = \sum_{n=-\infty}^{\infty} \alpha^{|n|} z^{-n} \quad (*)$$

when $n \leq 0$, $|n| = -n$

when $n \geq 0$, $|n| = n$

$$(*) = \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^{n} z^{-n} \quad (**)$$

$$= \lim_{A\to-\infty} \sum_{n=A}^{-1} \left(\frac{1}{\alpha}\right)^n (z^{-1})^n$$

$$= \lim_{A\to-\infty} \sum_{n=A}^{-1} \left(\frac{1}{\alpha} z^{-1}\right)^n$$

$$= \lim_{A\to-\infty} \frac{\left(\frac{1}{\alpha}z^{-1}\right)^A - \left(\frac{1}{\alpha}z^{-1}\right)^0}{1 - \frac{1}{\alpha}z^{-1}}$$

$$= \frac{0 - 1}{1 - \frac{1}{\alpha}z^{-1}} = \frac{-1}{1 - \frac{1}{\alpha}z^{-1}} \cdot \frac{-\alpha z}{-\alpha z}$$

$$= \frac{\alpha z}{1 - \alpha z}$$, provided that;

$$\left|\frac{1}{\alpha} z^{-1}\right| > 1 \quad \left(\text{b/c it's } \lim A \to -\infty\right)$$

$$\left|\frac{1}{\alpha}\right|\left|\frac{1}{z}\right| > 1 \rightarrow \frac{1}{|\alpha|} > |z|$$

$$= \lim_{A\to\infty} \sum_{n=0}^{A} \left(\frac{\alpha}{z}\right)^n$$

$$= \lim_{A\to\infty} \frac{\left(\frac{\alpha}{z}\right)^0 - \left(\frac{\alpha}{z}\right)^{A+1}}{1 - \alpha/z}$$

$$= \frac{1 - 0}{1 - \alpha z^{-1}} = \frac{1}{1 - \alpha z^{-1}},$$

provided that

$$\left|\frac{\alpha}{z}\right| < 1$$

$$|\alpha| < |z|$$

$\longrightarrow$

So $(**) = \underbrace{\dfrac{\alpha z}{1-\alpha z}}_{|z|<\frac{1}{|\alpha|}} + \underbrace{\dfrac{1}{1-\alpha z^{-1}}}_{|z|>|\alpha|}$

$$= \frac{\alpha z(1-\alpha z^{-1}) + (1-\alpha z)}{(1-\alpha z)(1-\alpha z^{-1})}, \quad |\alpha| < |z| < \frac{1}{|\alpha|}$$

$$= \frac{\alpha z - \alpha^2 + 1 - \alpha z}{(1-\alpha z)(1-\alpha z^{-1})} = \frac{1-\alpha^2}{(1-\alpha z)(1-\alpha z^{-1})} \cdot \frac{z}{z}$$

$$\boxed{X_a(z) = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}, \quad |\alpha| < |z| < \frac{1}{|\alpha|}}$$

(p/z-plot follows method II)

Method II : Table

$$x_a[n] = \alpha^{|n|} = \underset{(\text{for } n \geq 0)}{\alpha^n u[n]} + \underset{(\text{for } n \leq -1)}{\alpha^{-n} u[-n-1]}$$

$$= \alpha^n u[n] - -\left(\tfrac{1}{\alpha}\right)^n u[-n-1]$$

let $\beta = \frac{1}{\alpha}$

$$= \alpha^n u[n] - -\beta^n u[-n-1]$$

$\longrightarrow$

Table:

$$X_a(z) = \underbrace{\frac{1}{1-\alpha z^{-1}}}_{|z|>|\alpha|} - \underbrace{\frac{1}{1-\beta z^{-1}}}_{|z|<|\beta| = \left|\frac{1}{\alpha}\right|}$$

$$= \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\frac{1}{\alpha} z^{-1}} \cdot \frac{-\alpha z}{-\alpha z}, \quad |\alpha| < |z| < \frac{1}{|\alpha|}$$

$$= \frac{1}{1-\alpha z^{-1}} - \frac{-\alpha z}{-\alpha z + 1}$$

$$= \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z}{1-\alpha z}$$

$$= \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}, \quad |\alpha| < |z| < \frac{1}{|\alpha|}$$

(as on the previous page)

→ There is a zero at $z=0$.

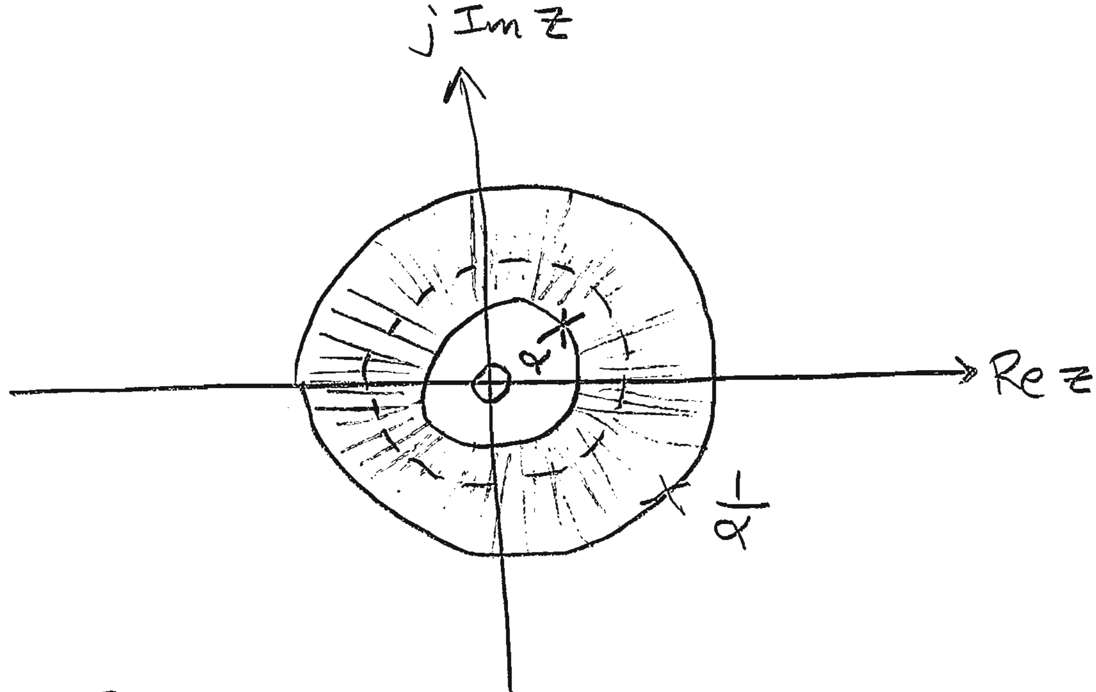
→ There is a pole at $z=\frac{1}{\alpha}$. It lies on a circle of radius $\frac{1}{|\alpha|}$, which is > 1 (since we were given that $0 \le |\alpha| < 1$).

→ There is a pole at $z=\alpha$. It lies on a circle of radius $|\alpha|$, which is < 1.

→ There is another zero @ $z=\infty$, but we don't show it in the pz-plot. ⟶

- We can't give an "exact" pz-plot because we don't know what α is, besides the given fact that $|\alpha| < 1$.

- But the pz-plot has to look something like this:

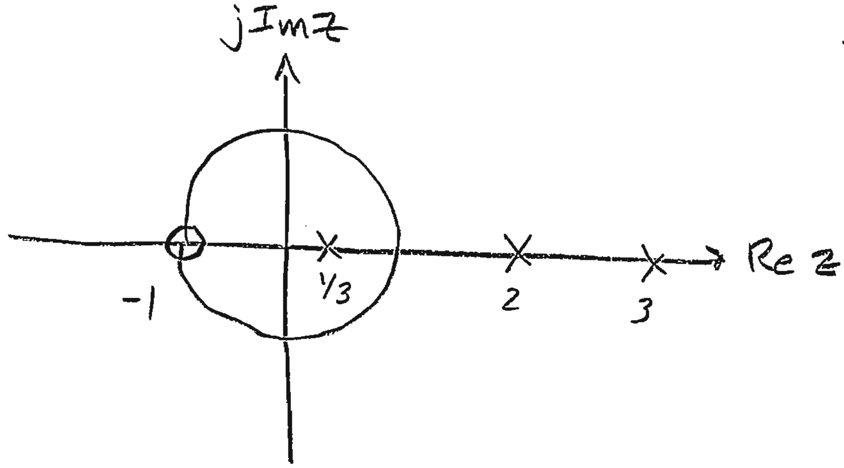


NOTE:

two-sided signal $\leftrightarrow$ annular ROC

3.4)

3.4-1



a) Given: $X(e^{j\omega})$ exists.

→ The ROC of $X(z)$ must contain the unit circle.

→ ROC must be $\frac{1}{3} < |z| < 2$

→ annular ROC → $x[n]$ must be two sided

b) The number of two-sided sequences (i.e., signals) is given by the number of annular ROCs, that are possible.

There are two: $\frac{1}{3} < |z| < 2$ and $2 < |z| < 3$.

$$\boxed{\Rightarrow \text{TWO possible two-sided sequences}}$$

c) Can $x[n]$ be both causal and stable?

$\boxed{\text{NO}}$

→ for $x[n]$ to be stable (i.e., $x[n] \in \ell_1(\mathbb{Z})$), the ROC must include the unit circle.

→ for $x[n]$ to be causal, the ROC must be exterior to the largest pole.

⇒ There is a pole at $z=3$

⇒ So these two conditions cannot be met simultaneously.

3.7) $x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}$$

a) Table:

$$u[-n-1] \longleftrightarrow \frac{-1}{1-z^{-1}}, \quad |z| < 1$$

$$\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-z^{-1}}$$

$$= \frac{(1-z^{-1}) - (1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$= \frac{1-z^{-1}-1+\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$= \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}, \quad \frac{1}{2} < |z| < 1$$

$\longrightarrow$

- The ROC of $Y(z)$ is not given.
- However, we are told that the system is causal. So the ROC of $H(z)$ must be exterior to the largest pole, whatever that turns out to be.
- So we have

$$H(z) = \frac{Y(z)}{X(z)} = Y(z)\left[X(z)\right]^{-1}$$

$$= \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} \cdot \frac{(1-\frac{1}{2}z^{-1})(1-z^{-1})}{-1/2\, z^{-1}}$$

$$= \frac{1-z^{-1}}{1+z^{-1}} \cdot \frac{z}{z} = \frac{z-1}{z+1}$$

→ one zero @ $z=1$
→ one pole @ $z=-1$
→ ROC must be $|z|>1$.

$$\boxed{H(z) = \frac{1-z^{-1}}{1+z^{-1}}, \quad |z|>1}$$

b) What is the ROC of $Y(z)$?

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} \cdot \frac{z^2}{z^2}$$

$$= \frac{-\frac{1}{2}z}{(z-\frac{1}{2})(z+1)}$$

poles: $\frac{1}{2}, -1$

zeros: $0, \infty$

The possible ROCs are:

$$|z| < \frac{1}{2}$$

$$\frac{1}{2} < |z| < 1$$

$$|z| > 1$$

- The ROC of $Y(z)$ must generally be the intersection of the ROC of $H(z)$ with the ROC of $X(z)$,
  - but it could be larger than that if there are pole-zero cancellations.
- From part (a), we know that the ROC of $X(z)$ is $\frac{1}{2} < |z| < 1$.
- From part (b), we know that the ROC of $H(z)$ is $|z| > 1$.

- So at first it seems like the intersectian must be the empty set.

- But the pole @ $z=1$ in $X(z)$ is cancelled by the zero @ $z=1$ in $H(z)$

- This <u>removes</u> the constraint for $|z|<1$ in the ROC of $X(z)$.

- So the ROC of $Y(z)$ is given by:

$$ROC = \{|z| > \tfrac{1}{2}\} \cap \{|z| > 1\} = |z| > 1$$

$$\boxed{\text{ROC of } Y(z): \ |z| > 1}$$

c) Find $y[n]$.

$\boxed{3.7-5}$

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}, \quad |z|>1$$

$$= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+z^{-1}}$$

$$A = \left.\frac{-\frac{1}{2}\theta}{1+\theta}\right|_{\theta=2} = \frac{-1}{3}$$

$$B = \left.\frac{-\frac{1}{2}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-1} = \frac{1/2}{1+1/2} = \frac{1/2}{3/2} = \frac{1}{3}$$

$$Y(z) = \underbrace{\frac{1/3}{1+z^{-1}}}_{|z|>1} - \underbrace{\frac{1/3}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}$$

$\rightarrow$ use $\alpha=-1$ for the first term and $\alpha=1/2$ for the second term

$$\boxed{y[n] = \frac{1}{3}(-1)^n u[n] - \frac{1}{3}\left(\frac{1}{2}\right)^n u[n]}$$

3.9) - The system H is causal

$\boxed{3.9-1}$

- The Transfer function is

$$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$$

a) Find the ROC of $H(z)$.

- because the system is causal, the ROC must be exterior to the largest pole.
- The poles are at $z=\frac{1}{2}$ and $z=-\frac{1}{4}$

$$\boxed{\text{ROC of } H(z):\ |z| > \frac{1}{2}}$$

b) Is the system stable?

$$\boxed{\text{YES, because the ROC includes the unit circle}}$$

c) What $X(z)$ will make the output

$$y[n] = -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3} 2^n u[-n-1] \quad ?$$

$\boxed{3.9\text{-}2}$

Table: $\left(-\frac{1}{4}\right)^n u[n] \longleftrightarrow \frac{1}{1+\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$

$-2^n u[-n-1] \longleftrightarrow \frac{1}{1-2z^{-1}}, \quad |z| < 2$

$$Y(z) = \frac{4/3}{1-2z^{-1}} - \frac{1/3}{1+\frac{1}{4}z^{-1}}, \quad \frac{1}{4} < |z| < 2$$

$$= \frac{\frac{4}{3}\left(1+\frac{1}{4}z^{-1}\right) - \frac{1}{3}\left(1-2z^{-1}\right)}{\left(1-2z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

$$= \frac{\frac{4}{3} + \frac{1}{3}z^{-1} - \frac{1}{3} + \frac{2}{3}z^{-1}}{\left(1-2z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

$$= \frac{1+z^{-1}}{\left(1-2z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}, \quad \frac{1}{4} < |z| < 2$$

Now, $Y(z) = X(z)H(z)$

So

$$X(z) = \frac{Y(z)}{H(z)} = Y(z) \cdot \frac{1}{H(z)}$$

$\longrightarrow$

$\boxed{3.9\text{-}3}$

$$X(z) = \frac{1+z^{-1}}{(1-2z^{-1})(1+\frac{1}{4}z^{-1})} \cdot \frac{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}{1+z^{-1}}$$

$$= \frac{1-\frac{1}{2}z^{-1}}{1-2z^{-1}}$$

- There is one pole at $z=2$
- The possible ROC's are $|z|<2$ and $|z|>2$
- We know that the ROC of $Y(z)$, which is $\frac{1}{4}<|z|<2$, must be the intersection of the ROC of $X(z)$ with the ROC of $H(z)$.
- The ROC of $H(z)$ is $\frac{1}{2}<|z|$.

$\Rightarrow$ So the ROC of $X(z)$ has to be $|z|<2$.

$$\boxed{X(z) = \frac{1-\frac{1}{2}z^{-1}}{1-2z^{-1}}, \quad |z|<2}$$

d) Find $h[n]$.

$$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}, \quad |z| > \frac{1}{2}$$

$$= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{4}z^{-1}}$$

$$A = \left.\frac{1+\theta}{1+\frac{1}{4}\theta}\right|_{\theta=2} = \frac{3}{1+\frac{1}{2}} = \frac{3}{3/2} = \frac{1}{1/2} = 2$$

$$B = \left.\frac{1+\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-4} = \frac{1-4}{1+2} = \frac{-3}{3} = -1$$

$$H(z) = \underbrace{\frac{2}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} - \underbrace{\frac{1}{1+\frac{1}{4}z^{-1}}}_{|z|>\frac{1}{4}}$$

Table: $\boxed{h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]}$

3.16) $x[n] = \left(\frac{1}{3}\right)^n u[n] + 2^n u[-n-1]$ $\boxed{3.16\text{-}1}$

$y[n] = 5\left(\frac{1}{3}\right)^n u[n] - 5\left(\frac{2}{3}\right)^n u[n]$

a) Find $H(z)$, ROC, and give p/z plot.

Table: $X(z) = \underbrace{\frac{1}{1-\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} - \underbrace{\frac{1}{1-2z^{-1}}}_{|z|<2}$

$= \frac{(1-2z^{-1}) - (1-\frac{1}{3}z^{-1})}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}, \quad \frac{1}{3} < |z| < 2$

$= \frac{1-2z^{-1}-1+\frac{1}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$

$= \frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}, \quad \frac{1}{3} < |z| < 2$

Table: $Y(z) = \underbrace{\frac{5}{1-\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} - \underbrace{\frac{5}{1-\frac{2}{3}z^{-1}}}_{|z|>\frac{2}{3}}$

$= \frac{5(1-\frac{2}{3}z^{-1}) - 5(1-\frac{1}{3}z^{-1})}{(1-\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})}, \quad |z| > \frac{2}{3}$

$\longrightarrow$

$$\ldots \; Y(z) = \frac{5 - \frac{10}{3}z^{-1} - 5 + \frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})}, \quad |z| > \frac{2}{3}$$

$$= \frac{-5/3\, z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})}, \quad |z| > \frac{2}{3}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-5/3\, z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})} \cdot \frac{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}{-5/3\, z^{-1}}$$

$$= \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}}$$

possible ROCs:

$|z| < 2/3 \qquad |z| > 2/3$

- In general, the ROC of $Y(z)$, which is $|z| > \frac{2}{3}$, must be the intersection of the ROC of $H(z)$ with the ROC of $X(z)$. But it could be larger than that if there is a pole-zero cancellation.
- We have:

$$\{\text{ROC of } Y(z)\} = \{\text{ROC of } X(z)\} \cap \{\text{ROC of } H(z)\}$$

$$\{|z| > \tfrac{2}{3}\} = \{\tfrac{1}{3} < |z| < 2\} \cap \{\text{ROC of } H(z)\}$$

→ But there is a pole-zero cancellation at $z = 2$.

→ So this becomes

$$\{|z| > \tfrac{2}{3}\} = \{|z| > \tfrac{1}{3}\} \cap \{\text{ROC of } H(z)\}$$

⟶

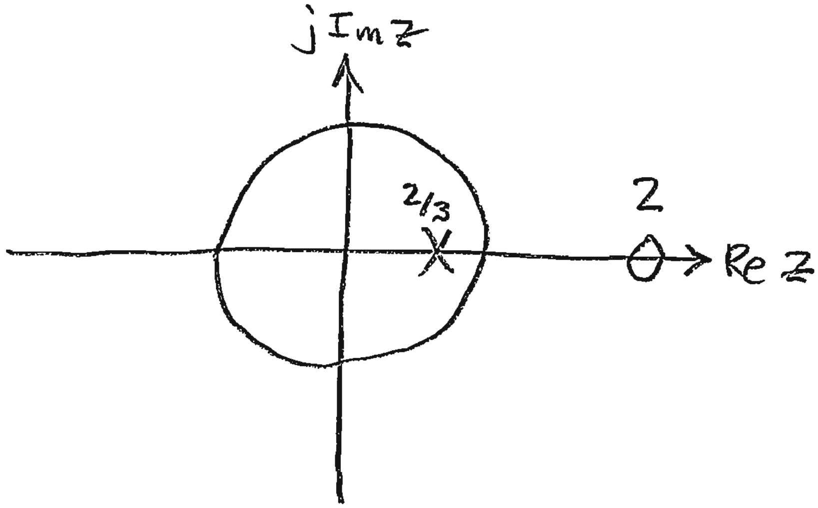
- of the two available choices for ROC of $H(z)$, $\boxed{3.16\text{-}3}$
which are $|z| < \frac{2}{3}$ or $|z| > 2/3$, it <u>must</u> be:

ROC of $H(z)$: $|z| > 2/3$.

- So $H(z) = \dfrac{1 - 2z^{-1}}{1 - \frac{2}{3}z^{-1}} = \dfrac{z - 2}{z - 2/3}$, $|z| > 2/3$

→ There is one zero @ $z = 2$
and one pole @ $z = 2/3$

$$\boxed{H(z) = \frac{1 - 2z^{-1}}{1 - \frac{2}{3}z^{-1}}, \quad |z| > 2/3}$$



b) Find $h[n]$.

3.16-4

$$H(z) = \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}} \quad , \ |z| > 2/3$$

$$= \underbrace{\frac{1}{1-\frac{2}{3}z^{-1}}}_{|z|>2/3} - \underbrace{\frac{2z^{-1}}{1-\frac{2}{3}z^{-1}}}_{|z|>2/3}$$

Table: $\frac{1}{1-\frac{2}{3}z^{-1}}, \ |z|>2/3 \longleftrightarrow \left(\frac{2}{3}\right)^n u[n]$

Table: $\frac{-2}{1-\frac{2}{3}z^{-1}}, \ |z|>2/3 \longleftrightarrow -2\left(\frac{2}{3}\right)^n u[n]$

Time-Shift property: $\frac{-2z^{-1}}{1-\frac{2}{3}z^{-1}}, \ |z|>\frac{2}{3} \longleftrightarrow -2\left(\frac{2}{3}\right)^{n-1} u[n-1]$

$$h[n] = \left(\tfrac{2}{3}\right)^n u[n] - 2\left(\tfrac{2}{3}\right)^{n-1} u[n-1]$$

$$= \left(\tfrac{2}{3}\right)^n u[n] - 2\left(\tfrac{2}{3}\right)^{-1}\left(\tfrac{2}{3}\right)^n u[n-1]$$

$$= \left(\tfrac{2}{3}\right)^n u[n] - 2\cdot\tfrac{3}{2}\left(\tfrac{2}{3}\right)^n u[n-1]$$

$$= \left(\tfrac{2}{3}\right)^n u[n] - 3\left(\tfrac{2}{3}\right)^n u[n-1]$$

$$\boxed{h[n] = \left(\tfrac{2}{3}\right)^n \left\{u[n] - 3u[n-1]\right\}}$$

c) Find the I/O relation:

$$H(z) = \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}} = \frac{Y(z)}{X(z)}$$

$$\left[1-\tfrac{2}{3}z^{-1}\right]Y(z) = \left[1-2z^{-1}\right]X(z)$$

$$Y(z) - \tfrac{2}{3}z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$$

$\mathcal{Z}^{-1}$:

$$\boxed{y[n] - \tfrac{2}{3}y[n-1] = x[n] - 2x[n-1]}$$

d) Is the system stable? causal?

Stable: YES because the ROC of $H(z)$ includes the unit circle.

Causal: YES because $h[n] = 0 \quad \forall\, n<0$.

3.21) $$H(z) = \frac{4 + \frac{1}{4}z^{-1} - \frac{1}{2}z^{-2}}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})}$$

→ we are given that the system H is causal.

a) Find the ROC of $H(z)$.

- There are poles @ $z = \frac{1}{4}$ and $z = -\frac{1}{2}$.
- The possible ROCs are:

$$|z| < \frac{1}{4} \ ; \quad \frac{1}{4} < |z| < \frac{1}{2} \ ; \quad |z| > \frac{1}{2}$$

- Because the system is causal, the ROC must be exterior to the largest pole:

$$\boxed{\text{ROC of } H(z): \ |z| > \frac{1}{2}}$$

b) Is the system stable?

<u>Yes</u>. Because the ROC of $H(z)$ includes the unit circle.

c) Find the I/O relation:

$$H(z) = \frac{4 + \frac{1}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{2}z^{-1} - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$= \frac{4 + \frac{1}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} = \frac{Y(z)}{X(z)}$$

⟶

Cross multiplying, we obtain:

$$Y(z)\left[1+\tfrac{1}{4}z^{-1}-\tfrac{1}{8}z^{-2}\right]=X(z)\left[4+\tfrac{1}{4}z^{-1}-\tfrac{1}{2}z^{-2}\right]$$

$$Y(z)+\tfrac{1}{4}z^{-1}Y(z)-\tfrac{1}{8}z^{-2}Y(z)=4X(z)+\tfrac{1}{4}z^{-1}X(z)-\tfrac{1}{2}z^{-2}X(z)$$

$\mathcal{Z}^{-1}$:

$$\boxed{y[n]+\tfrac{1}{4}y[n-1]-\tfrac{1}{8}y[n-2]=4x[n]+\tfrac{1}{4}x[n-1]-\tfrac{1}{2}x[n-2]}$$

d) Find $h[n]$.

→ Since the order of the numerator is <u>not</u> strictly less than the order of the denominator, $H(z)$ is an improper fraction.

→ This means that we can <u>not</u> apply partial fractions directly to $H(z)$. We must perform long division first to clear the improper fraction, then apply partial fractions to the remainder (which will then be a <u>proper fraction</u>.

dividend = numerator = $4+\tfrac{1}{4}z^{-1}-\tfrac{1}{2}z^{-2}$

divisor = denominator = $1+\tfrac{1}{4}z^{-1}-\tfrac{1}{8}z^{-2}$

→

$$\begin{array}{r|l} & \quad 4 \;\leftarrow \text{Quotient} \\ -\frac{1}{8}z^{-2}+\frac{1}{4}z^{-1}+1 & -\frac{1}{2}z^{-2}+\frac{1}{4}z^{-1}+4 \\ & -\frac{1}{2}z^{-2}+z^{-1}+4 \\ \hline & \quad -\frac{3}{4}z^{-1} \;\leftarrow \text{Remainder} \end{array}$$

$$H(z) = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

$$= 4 - \frac{\frac{3}{4}z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

$$= 4 - \frac{3/4\, z^{-1}}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})} \qquad (*)$$

- Note that the remainder is now a proper fraction
  → Apply PFE to it:

$$\frac{3/4\,\theta}{(1-\frac{1}{4}\theta)(1+\frac{1}{2}\theta)} = \frac{A}{1-\frac{1}{4}\theta} + \frac{B}{1+\frac{1}{2}\theta}$$

$$A = \left.\frac{3/4\,\theta}{1+\frac{1}{2}\theta}\right|_{\theta=4} = \frac{\frac{3}{4}\cdot 4}{1+\frac{1}{2}\cdot 4} = \frac{3}{1+2} = \frac{3}{3} = 1$$

$$B = \left.\frac{3/4\,\theta}{1-\frac{1}{4}\theta}\right|_{\theta=-2} = \frac{\frac{3}{4}\cdot(-2)}{1-\frac{1}{4}(-2)} = \frac{-3/2}{1+1/2} = \frac{-3/2}{3/2} = -1$$

So $$\frac{3/4\, z^{-1}}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})} = \frac{1}{1-\frac{1}{4}z^{-1}} - \frac{1}{1+\frac{1}{2}z^{-1}}$$

$\longrightarrow$

Plugging this PFE back into (*), we obtain: $\boxed{3.21-4}$

$$H(z) = 4 - \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{1}{1+\frac{1}{2}z^{-1}}$$

- Since the system is causal the ROC of $H(z)$ is exterior... so the ROCs for the individual terms must all be exterior... so that their intersection is equal to the overall ROC.
- we get

$$H(z) = \underbrace{4}_{\text{all } z} - \underbrace{\frac{1}{1-\frac{1}{4}z^{-1}}}_{|z|>\frac{1}{4}} + \underbrace{\frac{1}{1+\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}$$

Table:

$$\boxed{h[n] = 4\delta[n] - \left(\tfrac{1}{4}\right)^n u[n] + \left(-\tfrac{1}{2}\right)^n u[n]}$$

e) Find $Y(z)$ when $x[n] = u[-n-1]$

Table: $X(z) = \frac{-1}{1-z^{-1}}$, $|z| < 1$

$\longrightarrow$

$$Y(z) = X(z)H(z)$$

$$= \frac{-4 - \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2}}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})(1-z^{-1})}$$

ROC: $\{|z| > \frac{1}{2}\} \cap \{|z| < 1\} = \frac{1}{2} < |z| < 1$

$$\boxed{Y(z) = \frac{-4 - \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2}}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})(1-z^{-1})}, \quad \frac{1}{2} < |z| < 1}$$

f) Find $y[n]$.

- Since $Y(z)$ is a proper fraction, we can apply partial fractions directly.

$$\frac{-4 - \frac{1}{4}\theta + \frac{1}{2}\theta^2}{(1-\frac{1}{4}\theta)(1+\frac{1}{2}\theta)(1-\theta)} = \frac{A}{1-\frac{1}{4}\theta} + \frac{B}{1+\frac{1}{2}\theta} + \frac{C}{1-\theta}$$

$$A = \left.\frac{-4 - \frac{1}{4}\theta + \frac{1}{2}\theta^2}{(1+\frac{1}{2}\theta)(1-\theta)}\right|_{\theta=4} = \frac{-4 - \frac{1}{4}(4) + \frac{1}{2}(4)^2}{(1+\frac{1}{2}\cdot 4)(1-4)}$$

$$= \frac{-4 - 1 + \frac{1}{2}(16)}{(1+2)(-3)} = \frac{-5+8}{3(-3)} = \frac{3}{3(-3)} = -\frac{1}{3}$$

$\longrightarrow$

$\boxed{3.21\text{-}6}$

$$B = \left.\frac{-4-\frac{1}{4}\theta+\frac{1}{2}\theta^2}{(1-\frac{1}{4}\theta)(1-\theta)}\right|_{\theta=-2} = \frac{-4-\frac{1}{4}(-2)+\frac{1}{2}(-2)^2}{(1-\frac{1}{4}(-2))(1-(-2))}$$

$$= \frac{-4+\frac{1}{2}+\frac{1}{2}(4)}{(1+\frac{1}{2})(1+2)} = \frac{-4+\frac{1}{2}+2}{(\frac{3}{2})(3)} = \frac{-2+\frac{1}{2}}{9/2}$$

$$= \frac{-3/2}{9/2} = -\frac{3}{9} = -\frac{1}{3} \quad //$$

$$C = \left.\frac{-4-\frac{1}{4}\theta+\frac{1}{2}\theta^2}{(1-\frac{1}{4}\theta)(1+\frac{1}{2}\theta)}\right|_{\theta=1} = \frac{-4-\frac{1}{4}+\frac{1}{2}}{(1-\frac{1}{4})(1+\frac{1}{2})} = \frac{-4+\frac{1}{4}}{(\frac{3}{4})(\frac{3}{2})}$$

$$= \frac{-15/4}{9/8} = -\frac{8}{9}\cdot\frac{15}{4} = -\frac{2\cdot\cancel{4}}{3\cdot\cancel{3}}\cdot\frac{\cancel{3}\cdot 5}{\cancel{4}}$$

$$= \frac{-10}{3} \quad //$$

$$Y(z) = \underbrace{\frac{-1/3}{1-\frac{1}{4}z^{-1}}}_{|z|>\frac{1}{4}} + \underbrace{\frac{-1/3}{1+\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{-10/3}{1-z^{-1}}}_{|z|<1}$$

Table:

$$\boxed{y[n] = -\frac{1}{3}\left(\frac{1}{4}\right)^n u[n] - \frac{1}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{10}{3}u[-n-1]}$$

3.29 a) We are asked to use the unilateral z-transform. $\boxed{3.29a-1}$

- For unilateral z-transforms, I will write a "scripty" uppercase letter with a subscript "u", as in

$$x[n] \overset{\mathcal{U}z}{\longleftrightarrow} \mathcal{X}_u(z) \qquad y[n] \overset{\mathcal{U}z}{\longleftrightarrow} \mathcal{Y}_u(z)$$

- For an $x[n]$ that is <u>zero</u> $\forall\, n<0$, the unilateral z-transform is the same as the bilateral z-transform. So we can use the "regular" z-transform table to obtain unilateral z-transforms for any $x[n]$ that is zero on the negative n's.

- But some of the properties are significantly different for the unilateral z-transform.

- One of these is the time shift property. The unilateral z-transform time shift property involves a <u>time domain</u> initial condition on the signal.

- Specifically, it says that:

<u>IF</u> $x[n] \overset{\mathcal{U}z}{\longleftrightarrow} \mathcal{X}_u(z)$

<u>THEN</u> $x[n-1] \overset{\mathcal{U}z}{\longleftrightarrow} z^{-1}\mathcal{X}_u(z) + \underbrace{x[-1]}$

This is the time domain value of $x[n]$ at $n=-1$.

$\longrightarrow$

Given: $y[n] + 3y[n-1] = x[n]$

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

$$y[-1] = 1$$

Find: $y[n]$

- Apply the $\mathcal{U}z$ transform to both sides of the I/O equation:

$$\mathcal{Y}_u(z) + 3\{z^{-1}\mathcal{Y}_u(z) + y[-1]\} = \mathcal{X}_u(z)$$

$$\mathcal{Y}_u(z) + 3z^{-1}\mathcal{Y}_u(z) + 3(1) = \mathcal{X}_u(z)$$

$$[1 + 3z^{-1}]\,\mathcal{Y}_u(z) + 3 = \mathcal{X}_u(z) \qquad (*)$$

Table: $\mathcal{X}_u(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1}}$ $\qquad (**)$

→ Since it is understood that the $\mathcal{U}z$ transform only applies to a right-sided signal, the ROC is <u>always</u> exterior to the largest pole. In this case, it's $|z| > \frac{1}{2}$. But since this is understood, you don't have to write it.

- For a $\mathcal{U}z$ transform, the ROC is always understood to be everything exterior to the largest pole.

⟶

Plugging (**) into (*), we get

$$[1+3z^{-1}]\,Y_u(z)+3=\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$[1+3z^{-1}]\,Y_u(z)=\frac{1}{1-\frac{1}{2}z^{-1}}-3$$

$$[1+3z^{-1}]\,Y_u(z)=\frac{1-3(1-\frac{1}{2}z^{-1})}{1-\frac{1}{2}z^{-1}}=\frac{1-3+\frac{3}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$=\frac{-2+\frac{3}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$Y_u(z)=\frac{-2+\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}$$

$$=\frac{A}{1-\frac{1}{2}z^{-1}}+\frac{B}{1+3z^{-1}}$$

$$A=\left.\frac{-2+\frac{3}{2}\theta}{1+3\theta}\right|_{\theta=2}=\frac{-2+\frac{3}{2}(2)}{1+3(2)}=\frac{-2+3}{1+6}=\frac{1}{7}$$

$$B=\left.\frac{-2+\frac{3}{2}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-\frac{1}{3}}=\frac{-2+\frac{3}{2}(-\frac{1}{3})}{1-\frac{1}{2}(-\frac{1}{3})}=\frac{-2-\frac{1}{2}}{1+\frac{1}{6}}$$

$$=\frac{-5/2}{7/6}=\frac{6}{7}\left(-\frac{5}{2}\right)=-\frac{15}{7}$$

$$Y_u(z)=\frac{1/7}{1-\frac{1}{2}z^{-1}}-\frac{15/7}{1+3z^{-1}}$$

Table:

$$\boxed{y[n]=\frac{1}{7}\left(\frac{1}{2}\right)^n u[n]-\frac{15}{7}(-3)^n u[n],\quad n\geq 0}$$

3.45) $x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$ $\boxed{3.45\text{-}1}$

$$= \left(\frac{1}{2}\right)^n u[n] - -2^n u[-n-1]$$

Table: $X(z) = \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} - \underbrace{\frac{1}{1-2z^{-1}}}_{|z|<2}$

$$= \frac{(1-2z^{-1}) - (1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \quad \frac{1}{2} < |z| < 2$$

$$= \frac{-2z^{-1} + \frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

$$= \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \quad \frac{1}{2} < |z| < 2$$

$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n]$

Table: $Y(z) = \underbrace{\frac{6}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} - \underbrace{\frac{6}{1-\frac{3}{4}z^{-1}}}_{|z|>\frac{3}{4}}$

$$= \frac{6(1-\frac{3}{4}z^{-1}) - 6(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})}, \quad |z| > \frac{3}{4}$$

$$= \frac{6 - \frac{18}{4}z^{-1} - 6 + \frac{6}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} \longrightarrow$$

$\boxed{3.45\text{-}2}$

$$\ldots\ Y(z) = \frac{-\frac{9}{2}z^{-1} + \frac{6}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} = \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})}, \quad |z| > 3/4$$

$$H(z) = \frac{Y(z)}{X(z)} = Y(z) \cdot \frac{1}{X(z)}$$

$$= \frac{-3/2\, z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} \cdot \frac{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}{-3/2\, z^{-1}}$$

$$= \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$$

Now, the question is: what is the ROC of $H(z)$?

- There are two choices:

$$|z| > \tfrac{3}{4} \quad \text{-or-} \quad |z| < \tfrac{3}{4}$$

- We know that the ROC of $Y(z)$, which is $|z| > \frac{3}{4}$, should be the <u>intersection</u> of the ROC of $H(z)$ with the ROC of $X(z)$.
- But the ROC of $X(z)$ is $\frac{1}{2} < |z| < 2$.

$\Rightarrow$ At first, it doesn't seem like there's any way to intersect this with the ROC of $H(z)$ (either choice) and get the ROC of $Y(z)$???

$\longrightarrow$

However, note that $X(z)$ has a pole at $z=2$ which is actually what is causing the troublesome part of the ROC: $\frac{1}{2} < \underbrace{|z| < 2}_{\text{This part is the problem.}}$

$\Rightarrow$ And $H(z)$ has a zero at $z=2$.
So when we compute $Y(z) = X(z)H(z)$, there is a pole-zero cancellation at $z=2$. ☆
$\Rightarrow$ This cancels the restriction of $|z|<2$ in the ROC of $X(z)$.

- So we have:

$$\{\text{ROC of } Y(z)\} = \{\text{ROC of } X(z) \underline{\text{without}}\ |z|<2\} \cap \{\text{ROC of } H(z)\}$$

$$\rightarrow \{|z| > \tfrac{3}{4}\} = \{|z| > \tfrac{1}{2}\} \cap \{\text{ROC of } H(z)\}$$

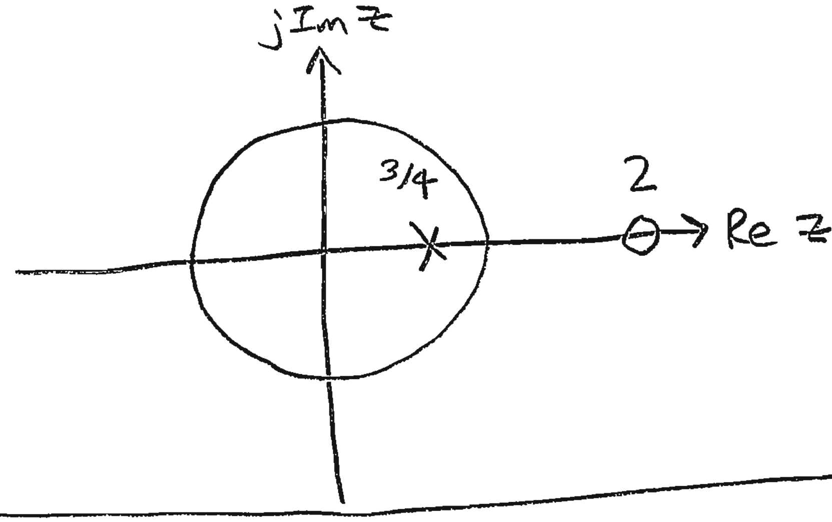
$\Rightarrow$ The ROC of $H(z)$ must be $|z| > \frac{3}{4}$

$\longrightarrow$

Now, $H(z)$ has one zero at $z=2$ and one pole at $z=3/4$.

So, the answer for part (a) is:

$$H(z) = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \quad |z| > \frac{3}{4}$$



b) Find $h[n]$:

$$H(z) = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \quad |z| > \frac{3}{4}$$

$$= \frac{1}{1-\frac{3}{4}z^{-1}} - \frac{2z^{-1}}{1-\frac{3}{4}z^{-1}}, \quad |z| > \frac{3}{4}$$

Table + time shift property:

$$h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]$$

$\longrightarrow$

$$\ldots h[n] = \left(\tfrac{3}{4}\right)^n u[n] - 2\left(\tfrac{3}{4}\right)^{-1}\left(\tfrac{3}{4}\right)^n u[n-1]$$

$$= \left(\tfrac{3}{4}\right)^n u[n] - 2\left(\tfrac{4}{3}\right)\left(\tfrac{3}{4}\right)^n u[n-1]$$

$$= \left(\tfrac{3}{4}\right)^n \left[u[n] - \tfrac{8}{3}u[n-1]\right]$$

$$\boxed{h[n] = \left(\tfrac{3}{4}\right)^n \left(u[n] - \tfrac{8}{3}u[n-1]\right)}$$

c) Find the I/O relation:

$$H(z) = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} = \frac{Y(z)}{X(z)}$$

$$\left[1-\tfrac{3}{4}z^{-1}\right]Y(z) = \left[1-2z^{-1}\right]X(z)$$

$$Y(z) - \tfrac{3}{4}z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$$

$$\mathcal{Z}^{-1}: \quad \boxed{y[n] - \tfrac{3}{4}y[n-1] = x[n] - 2x[n-1]}$$

d) The system IS STABLE because the ROC of $H(z)$ includes the unit circle.

The system IS CAUSAL because $h[n]=0 \;\forall\, n<0$.