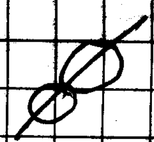
Another problem, discovered ~ 8/5 :

The previous filterbank design is in Book XI p. 102.

- The octave BW, B, is not in octaves.
- adjacent filters on a ray intersect @ η-peaks,

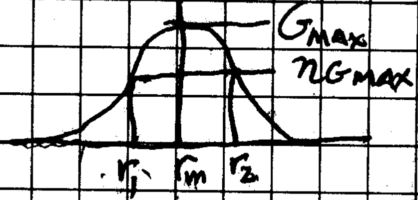


, but this means there are holes between rays where all filters are below η-peak. This could very well explain why I am having so much trouble holding onto my tracks.

- I will also change σ to be on top in freq, bottom in space, consistent with AC.

8/14/94 REVISED DEFINITION OF OCTAVE BANDWIDTH

Here is the correct definition of radial-octave bandwidth for a filter, taken along the principle axis (if not unity aspect ratio):



η-peak radial Octave Bandwidth is $B = \log_2 \frac{r_2}{r_1}$ octaves

$r_2 = 2r_1$: $B = \log_2 \frac{2}{1} = 1$

$r_2 = 2^N r_1$: $B = \log_2 \frac{2^N}{1} = N$

8/14/94 DESIGN OF NEW WAVELET (continued over):

Unity Variance Baseband Filter: $h(x,y) = A\exp\left[-\frac{1}{4}(x^2+y^2)\right]$

$$\|h\|_{L^2} = 1 = \int_{\mathbb{R}^2} |h(x,y)|^2\,dx\,dy$$

$$= \int_{\mathbb{R}^2} A^2 \exp\left[-\tfrac{1}{2}(x^2+y^2)\right] dx\,dy = A^2 \int_{\mathbb{R}^2} \exp\left[-\tfrac{1}{2}x^2\right]\exp\left[-\tfrac{1}{2}y^2\right] dx\,dy$$

$$= A^2\left[\int_{\mathbb{R}} \exp\left[-\tfrac{1}{2}x^2\right]dx\right]^2 = A^2 2\pi$$

$$\Rightarrow A = \frac{1}{\sqrt{2\pi}}$$

$$\boxed{h(x,y) = \frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{4}(x^2+y^2)\right]}$$ Unity Variance Baseband Filter

Add scaling:

$$h_\sigma(x,y) = A_\sigma \exp\left[\frac{-1}{4\sigma^2}(x^2+y^2)\right]$$

$$\|h_\sigma\|_{L^2} = 1 = \int_{\mathbb{R}^2} |h_\sigma(x,y)|^2\,dx\,dy = A_\sigma^2 \int_{\mathbb{R}^2} \exp\left[\frac{-1}{2\sigma^2}(x^2+y^2)\right]dx\,dy$$

$$= A_\sigma^2\left[\int_{\mathbb{R}} \exp\left[\frac{-1}{2\sigma^2}x^2\right]dx\right]^2 = A_\sigma^2\left[2\pi\sigma^2\right]$$

$$\Rightarrow A_\sigma = \frac{1}{\sqrt{2\pi}\,\sigma}$$

$$\boxed{h_\sigma(x,y) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[\frac{-1}{4\sigma^2}(x^2+y^2)\right]}$$ Scaled Baseband Filter

Add Modulation (Translation):

$U_m, V_m \rightarrow \frac{\text{cycle}}{\text{unit}}$

$$g_m(x,y) = h_{\sigma_m}(x,y)\exp\left[j2\pi(u_m x + v_m y)\right]$$

$$\|g_m\|_{L^2} = 1 = \int_{\mathbb{R}^2} |g_m(x,y)|^2\,dx\,dy$$

$$= \int_{\mathbb{R}^2} \left|h_{\sigma_m}(x,y)\exp\left[j2\pi(u_m x+v_m y)\right]\right|^2 dx\,dy$$

$$= \int_{\mathbb{R}^2} |h_{\sigma_m}(x,y)|^2\,dx\,dy = 1.$$

$$\boxed{g_m(x,y) = \frac{1}{\sqrt{2\pi}\,\sigma_m}\exp\left[\frac{-1}{4\sigma_m^2}(x^2+y^2)\right]\exp\left[j2\pi(u_m x+v_m y)\right]}$$

Space Domain [illegible]

8/14/94 Find Fourier Transform:

$$G_m(u,v) = \int_{\mathbb{R}^2} g_m(x,y)\exp[-j2\pi(ux+vy)]\,dx\,dy$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}\sigma_m}\exp\left[\frac{-1}{4\sigma_m^2}(x^2+y^2)\right]\exp[j2\pi(u_m x+v_m y)]\exp[-j2\pi(ux+vy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_m}\left[\int_{\mathbb{R}}\exp\left\{-\frac{1}{4\sigma_m^2}x^2\right\}\exp\{j2\pi(u_m x)\}\exp\{-j2\pi ux\}\,dx\right]$$

$$\cdot\left[\int_{\mathbb{R}}\exp\left\{-\frac{1}{4\sigma_m^2}y^2\right\}\exp\{j2\pi(v_m y)\}\exp\{-j2\pi vy\}\,dy\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma_m}\left[\int_{\mathbb{R}}\exp\left\{\frac{-1}{4\sigma_m^2}x^2\right\}\exp\{-j2\pi(u-u_m)x\}\,dx\right]$$

$$\cdot\left[\int_{\mathbb{R}}\exp\left\{\frac{-1}{4\sigma_m^2}y^2\right\}\exp\{-j2\pi(v-v_m)y\}\,dy\right]$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma_m}\right)\left(\frac{1}{4\sigma_m^2\pi}\right)^{-1}\int_{\mathbb{R}}\left(\frac{1}{4\sigma_m^2\pi}\right)^{\frac{1}{2}}\exp\left\{\frac{-1}{4\sigma_m^2}x^2\right\}\exp[-j2\pi(u-u_m)x]\,dx$$

$$\cdot\int_{\mathbb{R}}\left(\frac{1}{4\sigma_m^2\pi}\right)^{\frac{1}{2}}\exp\left\{\frac{-1}{4\sigma_m^2}y^2\right\}\exp[-j2\pi(v-v_m)y]\,dy$$

NOTE: from E.O. Brigham: $\left(\frac{\alpha}{\pi}\right)^{1/2}e^{-\alpha t^2} \Longleftrightarrow \exp\left[\frac{-\pi^2 f^2}{\alpha}\right]$

$$G_m(u,v) = \left(\frac{1}{\cancel{\sqrt{2\pi}}\cancel{\sigma_m}}\right)2\,\cancel{\sqrt{2\pi}}\sqrt{2\pi}\,\cancel{\sigma_m}\sigma_m\exp\left[-4\sigma_m^2\pi^2(u-u_m)^2\right]\exp\left[-4\sigma_m^2\pi\right.$$

$$\boxed{G_m(u,v) = 2\sqrt{2\pi}\,\sigma_m\exp\left\{-4\sigma_m^2\pi^2\left[(u-u_m)^2+(v-v_m)^2\right]\right\}}$$

6/14/94 Design σ_m for a η-peak radial bandwidth of B octaves:

let $r_m = \sqrt{u_m^2 + v_m^2}$; $u_m = r_m \cos\theta_m$; $v_m = r_m \sin\theta_m$.

The Polar form of G_m is:

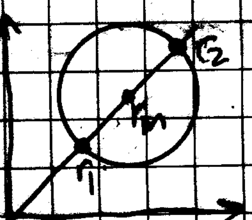
$$G_m(r,\theta) = 2\sqrt{2\pi}\,\sigma_m \exp\left(-4\pi^2\sigma_m^2\left[(r\cos\theta - r_m\cos\theta_m)^2 + (r\sin\theta - r_m\sin\theta_m)^2\right]\right)$$

Along the radial Axis of the filter:

$$G_m(r,\theta_m) = 2\sqrt{2\pi}\,\sigma_m \exp\left\{-4\pi^2\sigma_m^2\left[(r\cos\theta_m - r_m\cos\theta_m)^2 + (r\sin\theta_m - r_m\sin\theta_m)^2\right]\right\}$$

$$= 2\sqrt{2\pi}\,\sigma_m \exp\left\{-4\pi^2\sigma_m^2\left[(r-r_m)^2\cos^2\theta + (r-r_m)^2\sin^2\theta_m\right]\right\}$$

$$\boxed{G_m(r,\theta_m) = 2\sqrt{2\pi}\,\sigma_m \exp\left\{-4\pi^2\sigma_m^2 (r-r_m)^2\right\}}$$



Find r_1, r_2 for η-peak radial octave bandwidth $= B$:

$$\exp\left\{-4\pi^2\sigma_m^2 (r_2 - r_m)^2\right\} = \eta$$

$$-\ln\eta = 4\pi^2\sigma_m^2 (r_2 - r_m)^2$$

$$\sqrt{-\ln\eta} = 2\pi\sigma_m (r_2 - r_m)$$

$$r_2 = r_m + \frac{\sqrt{-\ln\eta}}{2\pi\sigma_m}$$

$$r_1 = r_m - \frac{\sqrt{-\ln\eta}}{2\pi\sigma_m}$$

$$\boxed{\sigma_m = \frac{2^B+1}{2^B-1}\,\frac{\sqrt{-\ln\eta}}{2\pi r_m}}$$

$$2^B = \frac{r_2}{r_1} = \frac{r_m + \frac{\sqrt{-\ln\eta}}{2\pi\sigma_m}}{r_m - \frac{\sqrt{-\ln\eta}}{2\pi\sigma_m}} = \frac{2\pi\sigma_m r_m + \sqrt{-\ln\eta}}{2\pi\sigma_m r_m - \sqrt{-\ln\eta}}$$

$$2^B 2\pi\sigma_m r_m - 2^B\sqrt{-\ln\eta} = 2\pi\sigma_m r_m + \sqrt{-\ln\eta}$$

$$(2^B - 1)\,2\pi\sigma_m r_m = (2^B + 1)\sqrt{-\ln\eta}$$

$$\boxed{\sigma_m 2\pi r_m = \frac{2^B+1}{2^B-1}\sqrt{-\ln\eta}}$$

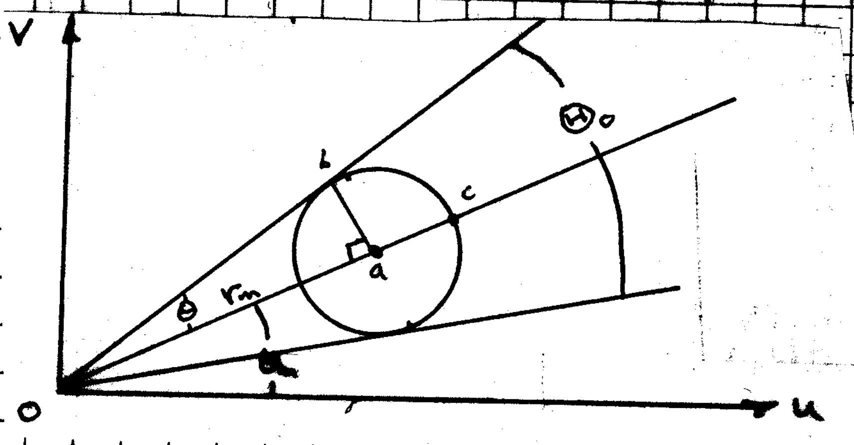
8/14/94...

The value of σ_m on p. 71 gives an η-peak radial bandwidth of:

$$\boxed{B = \log_2\left[\frac{\sigma_m 2\pi r_m + \sqrt{-\ln\eta}}{\sigma_m 2\pi r_m - \sqrt{-\ln\eta}}\right]}$$

~~Note~~: as $\sigma_m \rightarrow$ small, $r_m \rightarrow$ big, $G_m \rightarrow$ wide

Find the η-peak angular bandwidth, Θ_0, of G_m:



$$\overline{ab} = \overline{ac} = \frac{\sqrt{-\ln\eta}}{2\pi\sigma_m}$$

$$\tan\theta = \frac{\overline{ab}}{\overline{oa}} = \frac{\sqrt{-\ln\eta}}{\sigma_m 2\pi r_m}$$

$$\theta = \arctan\left[\frac{\sqrt{-\ln\eta}}{\sigma_m 2\pi r_m}\right]$$

$$\boxed{\Theta_0 = 2\arctan\left[\frac{\sqrt{-\ln\eta}}{\sigma_m 2\pi r_m}\right]}$$

4/94 Summary of ... [illegible]

$$g_m(x,y) = \frac{1}{\sqrt{2\pi}\sigma_m} \exp\left\{-\frac{1}{4\sigma_m^2}(x^2+y^2)\right\} \exp\{[illegible]\}$$

$$G_m(u,v) = 2\sqrt{2\pi}\,\sigma_m \exp\left\{-4\sigma_m^2\pi^2\left[(u-u_m)^2+(v-v_m)^2\right]\right\}$$

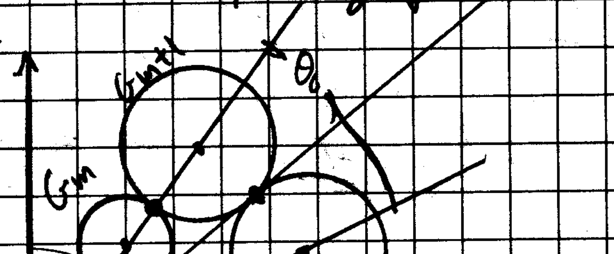
$$G_m(r,\theta_m) = 2\sqrt{2\pi}\,\sigma_m \exp\left\{-4\sigma_m^2\pi^2(r-r_m)^2\right\}$$

$$\sigma_m = \frac{2^B+1}{2^B-1}\,\frac{\sqrt{-\ln\eta}}{2\pi r_m} \qquad B = \log_2\left[\frac{\sigma_m 2\pi r_m + \sqrt{-\ln\eta}}{\sigma_m 2\pi r_m - \sqrt{-\ln\eta}}\right]$$

$$\Theta_0 = 2\arctan\left[\frac{\sqrt{-\ln\eta}}{\sigma_m 2\pi r_m}\right] = 2\arctan\sqrt{\gamma}$$

Design the filter spacing to give the "four corners" intersection, where 4 adjacent filters like the following:

- adjacent filters on the same ray intersect @ η-peak
- corresponding filters on adjacent rays intersect @ η-peak.



At the four intersections (Red dots) all filters are at η-peak.

Note: $\sigma_m = \sigma_n$

$\sigma_{m+1} = \sigma_{n+1}$

Find the center frequency spacing of G_m, G_{m+1}:

$$r_m + \frac{\sqrt{-\ln\eta}}{2\pi\sigma_m} = r_{m+1} - \frac{\sqrt{-\ln\eta}}{2\pi\sigma_{m+1}}$$

$\longrightarrow$