The problem with the filter tessellation on p. 76 is that between filters, there are places where no filter is at a peak. Find a denser tessellation where at least one filter is at a peak or higher at every point in the plane.

For this tessellation, the ratio $\frac{\theta_m}{\theta_n}$ is not specified... and the angular spacing $\theta$ is then found to produce the desired five-filter 2-peak intersection. Hence, the radial center frequencies along a ray follow a geometric progression with a specified common ratio. We call the common ratio "$R".
Hence \( R_{n+1} = R R_n \)

Let \( \alpha = \frac{\theta}{2} \).

Note: \( R_{n+1} = \frac{2^{\theta+1}}{2^{\theta-1}} \frac{\sqrt{2 \alpha n}}{2 \pi R_{n+1}} = \frac{2^{\theta+1} \sqrt{2 \alpha n}}{2^{\theta-1} 2 \pi R_{n+1}} \)

Let \( \alpha = R_{n+1} - R_n = R_n (R-1) \)

\[ \begin{align*}
\overrightarrow{BC} &= R_{n+1} - R_n = R_n (R-1) \\
\overrightarrow{BA} &= \frac{\sqrt{2 \alpha n}}{2 \pi R} = \frac{2^{\theta-1}}{2^{\theta+1}} R_n \\
\overrightarrow{CA} &= \frac{2^{\theta-1}}{2^{\theta+1}} (R_n) = \frac{2^{\theta-1}}{2^{\theta+1}} R R_n
\end{align*} \]

Let \( Y = \frac{(2^{\theta} - 1)^2}{(2^{\theta+1})^2} \) \quad \text{Let} \quad \beta = \angle CAB \)

**Law of Cosines for \( \angle CAB \):**

\[ \begin{align*}
R_n^2 (R-1)^2 &= Y R_n^2 + Y R^2 R_n^2 - 2 Y R R_n^2 \cos \beta \\
(R-1)^2 &= Y + Y R^2 - 2 Y R \cos \beta \\
2 Y R \cos \beta &= Y R^2 + Y - (R-1)^2 = Y R^2 - R^2 + 2 R - 1 + Y \\
2 Y R \cos \beta &= (Y-1)(R^2 + 1) + 2 R \\
\cos \beta &= \frac{(Y-1)(R^2 + 1) + 2 R}{2 Y R} \\
\beta &= \arccos \left( \frac{(Y-1)(R^2 + 1) + 2 R}{2 Y R} \right)
\end{align*} \]

Now \( \beta + \angle BAE = 180^\circ \Rightarrow \angle BAE = 180^\circ - \beta \Rightarrow \frac{\angle BAE}{2} = \frac{90^\circ}{2} = 45^\circ \cdot \frac{\theta}{2} \)

Also, \( \overrightarrow{BC} = R_n \sin \alpha = R_n \sin \frac{\theta}{2} = \overrightarrow{BA} \sin \frac{\alpha}{2} = \overrightarrow{BA} \sin \left(90^\circ - \frac{\theta}{2}\right) = \overrightarrow{BA} \cos \frac{\theta}{2} \)

\[ \begin{align*}
\sin \frac{\theta}{2} &= \overrightarrow{BA} \cos \frac{\theta}{2} \\
\Rightarrow \quad \sin \frac{\theta}{2} &= \overrightarrow{BA} \cos \frac{\theta}{2}
\end{align*} \]
Now \( B = \text{Quad I or Quad II}, \quad \Theta = \text{Quad I.} \)

So \( \frac{B}{2} \in \text{Quad I} \) and \( \frac{\Theta}{2} \in \text{Quad I}. \)

\[ \cos \frac{B}{2} = -\sqrt{\frac{1}{2} + \frac{1}{2} \cos B} \]

\[ \Rightarrow \frac{r_m \sin \frac{\Theta}{2}}{2} = \sqrt[3]{r_m \cos \frac{B}{2}} \]

\[ \sin \frac{\Theta}{2} = \sqrt{- \frac{1}{2} + \frac{1}{2} \cos B} = \sqrt{- \frac{1}{2} \left( 1 + \cos B \right)} \]

\[ = \left[ \frac{1}{2} \left( 1 + \frac{(y-1)(R^2+1) + 2R}{2Ry} \right) \right]^{1/2} \]

\[ = \left[ \frac{y + (y-1)(R^2+1) + 2R}{4R} \right]^{1/2} = \left[ \frac{2Ry + (y-1)(R^2+1) + 2R}{4R} \right]^{1/2} \]

\[ = \left[ \frac{(R^2+1)(y-1) + 2R(y+1)}{4R} \right]^{1/2} \]

\[ = \left[ 4R \right]^{-1/2} \left[ (R^2+1)(y-1) + 2R(y+1) \right]^{1/2} \]

\[ \Theta = 2 \arcsin \left\{ \left[ 4R \right]^{-1/2} \left[ (R^2+1)(y-1) + 2R(y+1) \right]^{1/2} \right\} \]

\[ r = 1.8 \quad B = 1 \quad \eta = \frac{1}{2} \quad \Rightarrow \quad \Theta = 20.6418^\circ \]

\[ r = 1.7 \quad B = 1 \quad \eta = \frac{1}{2} \quad \Rightarrow \quad \Theta = 25.0576^\circ \]
**Design of Filterbank**

\[ N_f = \text{number filters per ray.} \]

I will put the maximum radial center frequency as close to 0.5 cycles/sample as I can without exceeding it...

Although it would be possible to put more filters on rays with orientation \( \approx \pi/4 \), I prefer to have the same \( N_f \) of filters on every ray.

The first filter on the ray has radial center frequency \( R_0 \).

The radial center frequency of the \( N_f \text{th} \) filter is

\[ R_{N_f} = R_0^{N_f} \quad \text{if} \quad R_0 < 1/2 \text{ cycle/pixel} \]

\[ -\log_2 R > N_f - 1 + \log_2 R_0 \]

\[ \ln R \cdot \frac{\ln 2}{\ln R} \rightarrow N_f - 1 + \frac{\ln R_0}{\ln R} \]

\[ -\frac{\ln R_0 - \ln 2}{\ln R} > N_f - 1 \]

\[ +\frac{\ln R_0 - \ln 2}{\ln R} + 1 > N_f - 1 \]

\[ N_f = 1 \cdot \frac{\ln R_0 - \ln 2}{\ln R} + 1 \]

\[ N_f = \left[ \frac{\ln R_0 - \ln 2}{\ln R} \right] + 1 \]

\( R_0 \) is in cycles/pixel.

Check: \( N_f = 5 \) valid.
\( Nr = \) number of rays in half-plane

I rename \( \theta' \) on page 78, the angular spacing between the rays, \( \Lambda' \):

\[
\Lambda' = 2 \arcsin \left( \frac{\sqrt{[4R^2 - \frac{1}{2} \left( R^2 + 1 \right)(y - 1) + 2R(y + 1)^2]} + \sqrt{2}}{2} \right)
\]

\[
Y = \frac{(y - 1)^2}{(2y + 1)^2}
\]

\[
N_r = \left\lfloor \frac{\pi}{\Lambda'} \right\rfloor
\]

check: \( \Lambda' = 20.6416^\circ = 3.60257 \times 10^{-3} \text{ rad} \)

\( \Rightarrow \) \( Nr = 3 \) ✓

\( N_T = \) Total Number Filters in Filterbank = \( Nr \times N_r \).

Number the filters \( \phi = N_T - 1 \).

Filter \( m \) is on ray \( L_{\frac{m}{N_r}} \).

It is the \( m \mod N_r \) filter on that ray.

\[
\phi_m = \frac{\pi R^{\frac{1}{2}} - \frac{\pi}{2} L_{\frac{m}{N_r}}} {N_r} \text{ radians} = \text{angle (orientation)} \text{ of } m^{th} \text{ filter}
\]

\[
\phi_m + \frac{\pi}{2} L_{\frac{m}{N_r}} = \frac{\pi}{2} \text{ radians}
\]

\( \phi_m = \text{radial center frequency of } m^{th} \text{ filter} = \frac{R^{\frac{1}{2}}}{N_r} \text{ cycles/sample} \)

\( \phi_m = \cos \phi_m \text{ cycles/sample} \)

\( \phi_m = \text{horizontal center frequency of } m^{th} \text{ filter} = \phi_m \sin \phi_m \text{ cycles/sample} \)

\( \phi_m = \text{vertical center frequency of } m^{th} \text{ filter} \)
8/14/94... Finally, I correct the Postfilter design on p. 67 for the new σ in convention:

\[ P_{R}(x,y) = A \exp \left[ \frac{-1}{4\sigma^2} (x^2 + y^2) \right] \]

\[
\iint_{R} P_{R}(x,y) \, dx \, dy = 1 \Rightarrow \int_{R} \int_{R} P_{R}(x,y) \, dx \, dy = 1
\]

\[
= A \int_{R} \exp \left[ \frac{-1}{4\sigma^2} (x^2 + y^2) \right] dx \, dy
\]

\[
= A \int_{R} \exp \left[ \frac{-1}{4\sigma^2} x^2 \right] dx \int_{R} \exp \left[ \frac{-1}{4\sigma^2} y^2 \right] dy
\]

\[
= A \left[ \int_{R} \exp \left[ \frac{-1}{4\sigma^2} x^2 \right] dx \right]^2 = A \frac{\pi \sigma^2}{\pi \sigma^2}
\]

\[
A = \frac{1}{\pi \sigma^2}
\]

\[ P_{R}(x,y) = \frac{1}{\pi \sigma^2} \exp \left[ \frac{-1}{4\sigma^2} (x^2 + y^2) \right] \]

Let the postfilter sigma scaling factor be \( \kappa \).

the \( \kappa \)th postfilter has \( \sigma = \kappa \sigma_0 \)

\[ P_{R}(x,y) = \frac{1}{\pi \sigma_0^2} \exp \left[ \frac{-1}{4\sigma_0^2} (x^2 + y^2) \right] \]

Now I am ready... to summarize the entire filterbank design...
NEW FILTERBANK DESIGN

\[ R = \text{radial center freq of } l^{th} \text{ filter on each ray} \quad \text{cycles/pixel} \]
\[ R = \text{common ratio of filter radial center frequencies (dimensionless)} \]
\[ B = \text{radial octave bandwidth of filter in octaves} \]
\[ \gamma = \text{fraction of peak response defining bandwidth } B \quad \text{dimensionless} \]
\[ \mu = \text{postfilter frequency constant scaling factor (dimensionless)} \]

\[ N_f = \text{number of filters} = \left[ \frac{-\ln(2\gamma)}{2\ln R} \right] + 1 \]
\[ \theta^* = 2\arctan \sqrt{y} \]
\[ y = \frac{(2^\gamma - 1)^2}{(2^\gamma + 1)^2} \]
\[ \Delta = \text{angular spacing between rays} = 2\arcsin \left[ \frac{(4R^2 + 3(R^2 + 1)y - 1) + 2R(1+y)\gamma^2}{1 - \gamma^2} \right] \]

\[ N_r = \text{num rays} = \left\lfloor \frac{\pi}{\Delta} \right\rfloor \]
\[ N_r = \text{Tot Num filters} = N_f \cdot N_r \]

Filter \( m \) is on ray \( \frac{l}{N_r} \).

It is the \((m \mod N_r)\) th filter on this ray.

\[ \Theta_m = \frac{2\pi}{N_r} \left[ m + \frac{1}{2} \right] \text{ radians} \]
\[ \tau_m = R \left( m \mod N_r \right) \quad \text{cycles/pixel} \]
\[ \nu_m = \tau_m \cos \Theta_m \quad \text{cycle/pixel} \]
\[ \nu_m = \tau_m \sin \Theta_m \quad \text{cycle/pixel} \]
NEW FILTERBANK DESIGN CONT.

\[ O_m = \frac{\sqrt{\beta}}{2\pi r_m \sqrt{\gamma}} \]

\[ g_m(x, y) = \frac{1}{\sqrt{2\pi} \sigma_m} \exp \left[ -\frac{1}{2} \left( x^2 + y^2 \right) \right] \exp \left[ \pm i 2\pi m(x - u_m + \nu_m) \right] \]

\[ c_m(u, v) = 2\sqrt{2\pi} \sigma_m \exp \left[ -\frac{4\sigma_m^2 \pi^2}{2} \left( u^2 + v^2 \right) \right] \]

\[ \theta_m = \arctan \left( \frac{v}{u} \right) \]

\[ \sigma_m(x, y) = \frac{1}{\sqrt{2\pi} \sigma_m} \exp \left[ -\frac{1}{4\sigma_m^2 \pi^2} \left( x^2 + y^2 \right) \right] \]

Example: 256 x 256:

\[ c_m = 9.6 \text{ CPI} = 0.0375 \text{ cycles/pix} \]

\[ R = 1.8 \]

\[ B = 1.0 \]

\[ \eta = 0.5 \]

\[ K = 1.25 \]

\[ N_T = 5 \]

\[ \gamma = \frac{1}{9} \]

\[ \Lambda = 20.6418^\circ \]

\[ N_T = 8 \]

\[ N_T = 40 \]

\[ \theta = 38.9424^\circ \]
\( v_b = 3.6 \text{ cm/s} \)
\( R = 1.8 \)
\( B = 1.0 \)
\( \theta = 45^\circ \)

Made by PH.