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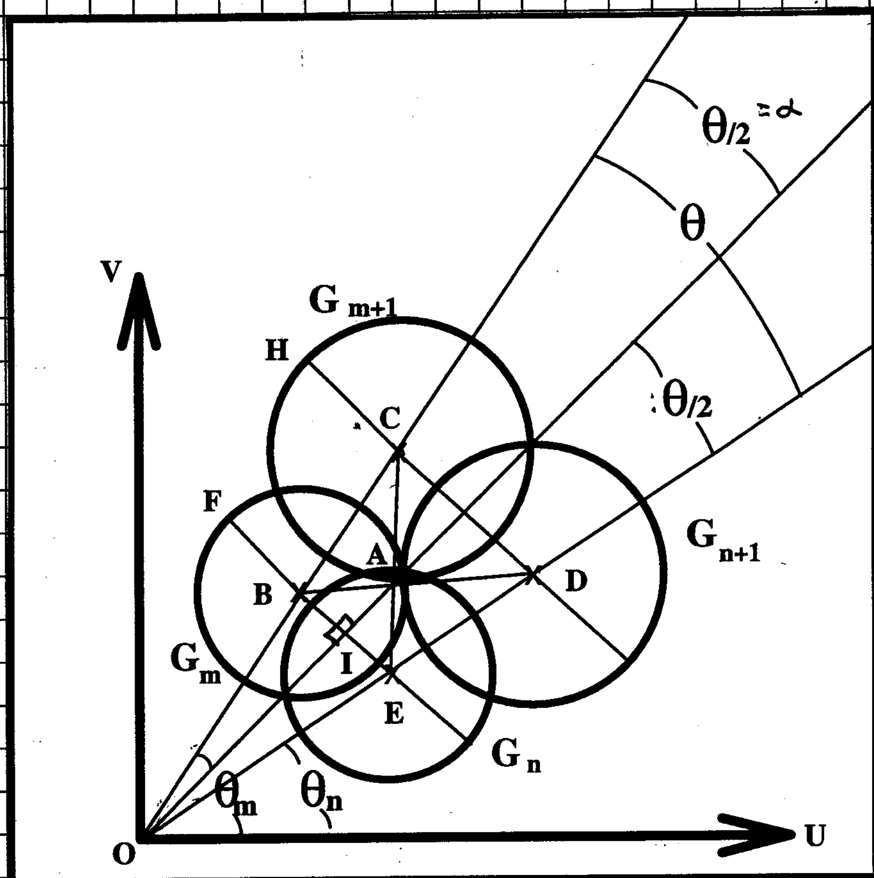
The problem with the filter tesselation on p. 76 is that, between filters, there are places where no filter is at η-peak.

Find a denser tesselation where at least one filter is at η-peak or higher at every point in the plane:



For this tesselation, the ratio $\frac{r_{m+1}}{r_m}$ is not designed, it is specified... and the angular spacing θ is then found to produce the desired four-filter η-peak intersection.

Hence, the radial center frequencies along a ray follow a geometric progression with a specified common ratio. We call the common ratio "R".

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Hence $r_{m+1} = R r_m$

Let $\alpha = \theta/2$.

Note: $\sigma_{m+1} = \frac{2^B+1}{2^B-1} \frac{\sqrt{-\ln \pi}}{2\pi r_{m+1}} = \frac{2^B+1}{2^B-1} \frac{\sqrt{-\ln \pi}}{2\pi r_m R}$

$$\overline{BC} = r_{m+1} - r_m = r_m (R-1)$$

$$\overline{BA} = \frac{\sqrt{-\ln \pi}}{2\pi\sigma_m} = \frac{2^B-1}{2^B+1} r_m$$

$$\overline{CA} = \frac{2^B-1}{2^B+1} r_{m+1} = \frac{2^B-1}{2^B+1} R r_m$$

Let $\gamma = \frac{(2^B-1)^2}{(2^B+1)^2}$ Let $\beta = \angle CAB$

LAW OF COSINES FOR $\angle CAB$:

$$r_m^2 (R-1)^2 = \gamma r_m^2 + \gamma R^2 r_m^2 - 2R\gamma r_m^2 \cos\beta$$

$$(R-1)^2 = \gamma + \gamma R^2 - 2R\gamma \cos\beta$$

$$2R\gamma\cos\beta = \gamma R^2 + \gamma - (R-1)^2 = \gamma R^2 - R^2 + 2R - 1 + \gamma$$

$$2R\gamma\cos\beta = (\gamma-1)(R^2+1) + 2R$$

$$\cos\beta = \frac{(\gamma-1)(R^2+1) + 2R}{2R\gamma}$$

$$\beta = \arccos\left[\frac{(\gamma-1)(R^2+1) + 2R}{2R\gamma}\right]$$

Now $\beta + \angle BAE = 180° \Rightarrow \angle BAE = 180° - \beta \Rightarrow \frac{\angle BAE}{2} = 90° - \beta/2$

Also, $\overline{BI} = r_m \sin\alpha = r_m \sin\frac{\theta}{2} = \overline{BA} \sin\frac{\angle BAE}{2} = \overline{BA}\sin(90 - \frac{\beta}{2}) = \overline{BA}\cos\frac{\beta}{2}$

so $r_m \sin\frac{\theta}{2} = \overline{BA}\cos\frac{\beta}{2}$ →

8/14/94 . . .

NOW $\beta \in$ Quad I or Quad II , $\theta \in$ Quad I .

So $\frac{\beta}{2} \in$ Quad I and $\theta/2 \in$ Quad I .

$$\Rightarrow \cos\frac{\beta}{2} = \sqrt{\frac{1}{2} + \frac{1}{2}\cos\beta}$$

$$\Rightarrow r_m \sin\frac{\theta}{2} = \sqrt{\gamma}\, r_m \cos\frac{\beta}{2}$$

$$\sin\frac{\theta}{2} = \sqrt{\gamma\left(\tfrac{1}{2} + \tfrac{1}{2}\cos\beta\right)} = \sqrt{\tfrac{\gamma}{2}(1+\cos\beta)}$$

$$= \left\{\frac{\gamma}{2}\left[1 + \frac{(\gamma-1)(R^2+1)+2R}{2R\gamma}\right]\right\}^{1/2}$$

$$= \left[\frac{\gamma}{2} + \frac{(\gamma-1)(R^2+1)+2R}{4R}\right]^{1/2} = \left[\frac{2R\gamma}{4R} + \frac{(\gamma-1)(R^2+1)+2R}{4R}\right]^{1/2}$$

$$= \left[\frac{(R^2+1)(\gamma-1)+2R(\gamma+1)}{4R}\right]^{1/2}$$

$$= [4R]^{-\frac{1}{2}}\left[(R^2+1)(\gamma-1)+2R(\gamma+1)\right]^{1/2}$$

$$\boxed{\theta = 2\arcsin\left\{[4R]^{-1/2}\left[(R^2+1)(\gamma-1)+2R(\gamma+1)\right]^{1/2}\right\}}$$

$R = 1.8 \quad B = 1 \quad \eta = \frac{1}{2} \quad \Rightarrow \quad \theta = 20.6418°$

$R = 1.7 \quad B = 1 \quad \eta = 1/2 \quad \Rightarrow \theta = 25.0576°$

8/14/94 DESIGN OF FILTERBANK:

N_f = number filters per ray.

I will put the maximum radial center frequency as close to 0.5 cycles/sample as I can w/out exceeding it...

Although it would be possible to put more filters on rays with orientation $\approx \pi/4$, I prefer to have the same # of filters on every ray.

The first filter on the ray has radial center frequency r_0.

The radial center frequency of the N_f th filter is

$$r_{N_f-1} = R^{N_f-1} r_0 < \tfrac{1}{2} \text{ cycle/pixel}$$

$$-\log_R 2 > N_f - 1 + \log_R r_0$$

$$\frac{-\ln 2}{\ln R} > N_f - 1 + \frac{\ln r_0}{\ln R}$$

$$-\frac{\ln r_0}{\ln R} - \frac{\ln 2}{\ln R} > N_f - 1$$

$$-\frac{\ln r_0}{\ln R} - \frac{\ln 2}{\ln R} + 1 > N_f$$

; N_f is an integer

$$N_f = \left\lfloor -\frac{\ln r_0}{\ln R} - \frac{\ln 2}{\ln R} + 1 \right\rfloor$$

$$N_f = \left\lfloor -\frac{\ln r_0}{\ln R} - \frac{\ln 2}{\ln R} \right\rfloor + 1$$

$$N_f = \left\lfloor \frac{-\ln 2 r_0}{\ln R} \right\rfloor + 1$$

check: $r_0 = 9/64 \pi$ (?) $\Rightarrow N_f = 5$ ✓

r_0 is in cycles/pix

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N_R = number of rays in half-plane.

I rename "θ" on page 78, the angular spacing between the rays, "Λ".

$$\Lambda = 2\arcsin\left\{\left[4R\right]^{-\frac{1}{2}}\left[(R^2+1)(\gamma-1)+2R(\gamma+1)\right]^{1/2}\right\}$$

$$\gamma = \frac{(2^B-1)^2}{(2^B+1)^2}$$

~~$N_R = \lfloor \frac{2\pi}{\Lambda} \rfloor$~~

$$\boxed{N_R = \left\lfloor \frac{\pi}{\Lambda} \right\rfloor}$$

check: $\Lambda = 20.6418° = 360.267 \times 10^{-3}$ rad

$\Rightarrow N_R = 8$ ✓

N_T = Total Number Filters in filterbank = $N_R N_f$.

Number the filters $\phi \to N_T - 1$.

filter m is on ray $\lfloor \frac{m}{N_f} \rfloor$

It is the m mod N_f filter on that ray.

θ_m = ~~$\frac{\pi}{N_R}\lfloor \frac{m}{N_R} \rfloor$ radians~~ $\Lambda/2 + \Lambda \lfloor \frac{m}{N_f} \rfloor - \pi/2$ radians = angle (orientation) of m^{th} filter

r_m = radial center frequency of m^{th} filter = $r_0 R^{(m \bmod N_f)}$ cycles/sample

u_m = horizontal center freq of m^{th} filter = $r_m \cos\theta_m$ cycles/sample

v_m = vertical center freq of m^{th} filter = $r_m \sin\theta_m$ cycles/sample.

8/14/94. Finally, I correct the postfilter design on p.67 for the new σ_m convention:

NEW POSTFILTER DESIGN

$$p_\sigma(x,y) = A\exp\left[\frac{-1}{4\sigma^2}(x^2+y^2)\right]$$

$$\|p_\sigma\|_{L^1} = 1 = \int_{\mathbb{R}^2} |p_\sigma(x,y)|\, dx\, dy$$

$$= A\int_{\mathbb{R}^2} \exp\left[\frac{-1}{4\sigma^2}(x^2+y^2)\right] dx\, dy$$

$$= A\int_{\mathbb{R}} \exp\left[\frac{-1}{4\sigma^2}x^2\right] dx \int_{\mathbb{R}} \exp\left[\frac{-1}{4\sigma^2}y^2\right] dy$$

$$= A\left[\int_{\mathbb{R}} \exp\left[\frac{-1}{4\sigma^2}x^2\right] dx\right]^2 = A\, 4\pi\sigma^2$$

$$A = \frac{1}{4\pi\sigma^2}$$

$$p_\sigma(x,y) = \frac{1}{4\pi\sigma^2}\exp\left[\frac{-1}{4\sigma^2}(x^2+y^2)\right]$$

Let the postfilter sigma scaling factor be κ.

the m^{th} postfilter has $\sigma = \kappa\sigma_m$

$$p_m(x,y) = \frac{1}{4\pi\kappa^2\sigma_m^2}\exp\left[\frac{-1}{4\kappa^2\sigma_m^2}(x^2+y^2)\right]$$

Now I am ready to summarize the entire filterbank design →

8/14/94

NEW FILTERBANK DESIGN

INPUTS

- r_0* = radial center freq of 1st filter on each ray ~~cycles/pixel~~
- R = common ratio of filter radial center frequencies (dimensionless)
- B = radial η-peak octave bandwidth of filters in octaves
- η = fraction of peak response defining bandwidth B (dimensionless)
- k = postfilter ~~space~~ frequency constant scaling factor (dimensionless)

* for the program, r_0 ~~is~~ cpi, and converted to cycles/pix immediately upon being read in.

N_f = num. filters/ray ~~rays~~ $= \left\lfloor \frac{-\ln 2r_0}{\ln R} \right\rfloor + 1$

$$\gamma = \frac{(2^B-1)^2}{(2^B+1)^2}$$

$$\Theta_0 = 2\arctan\sqrt{\gamma}$$

Λ = angular spacing between rays $= 2\arcsin\left[(4R)^{-\frac{1}{2}}\{(R^2+1)(\gamma-1)+2R(\gamma+1)\}^{\frac{1}{2}}\right]$

N_R = num rays $= \left\lfloor \frac{\pi}{\Lambda} \right\rfloor$

N_T = Tot Num filters $= N_R N_f$

filter m is on ray $\left\lfloor \frac{m}{N_f} \right\rfloor$

It is the $(m \bmod N_f)$ th filter on this ray

$\theta_m =$ ~~$\frac{2\pi}{N_R}\left\lfloor\frac{m}{N_f}\right\rfloor$ radians~~ $\Lambda\left\lfloor \frac{m}{N_f} \right\rfloor - \pi/2 + \frac{\Lambda}{2}$ radians

$r_m = r_0 R^{(m \bmod N_f)}$ cycles/pixel

$u_m = r_m \cos\theta_m$ cycles/pixel

$v_m = r_m \sin\theta_m$ cycles/pixel

→

NEW FILTERBANK DESIGN, CONT---

$$\sigma_m = \frac{\sqrt{-\ln \eta}}{2\pi r_m \sqrt{\gamma}}$$

$$g_m(x,y) = \frac{1}{\sqrt{2\pi}\,\sigma_m} \exp\left\{-\frac{1}{4\sigma_m^2}(x^2+y^2)\right\} \exp\left\{j2\pi(u_m x + v_m y)\right\}$$

$$G_m(u,v) = 2\sqrt{2\pi}\,\sigma_m \exp\left\{-4\sigma_m^2\pi^2\left[(u-u_m)^2+(v-v_m)^2\right]\right\}$$

$$G_m(r,\theta_m) = 2\sqrt{2\pi}\,\sigma_m \exp\left\{-4\sigma_m^2\pi^2(r-r_m)^2\right\}$$

see book XI p. 151

$$\Theta_m = 2\arctan\left[\frac{\sqrt{-\ln\eta}}{2\pi r_m \sigma_m}\right] = 2\arctan\sqrt{\gamma}$$

$$p_m(x,y) = \frac{1}{4\pi\kappa^2\sigma_m^2} \exp\left\{\frac{-1}{4\kappa^2\sigma_m^2}(x^2+y^2)\right\}$$

Example 256x256:

$r_0 = 9.6$ cpi $= 0.0375$ cycles/pix
$R = 1.8$
$B = 1.0$
$\eta = 0.5$
$\kappa = 1.25$

$N_F = 5$

$\gamma = \frac{1}{9}$

$\Lambda = 20.6418^\circ$

$N_R = 8$

$N_T = 40$

$\Theta_0 = 38.9424^\circ$

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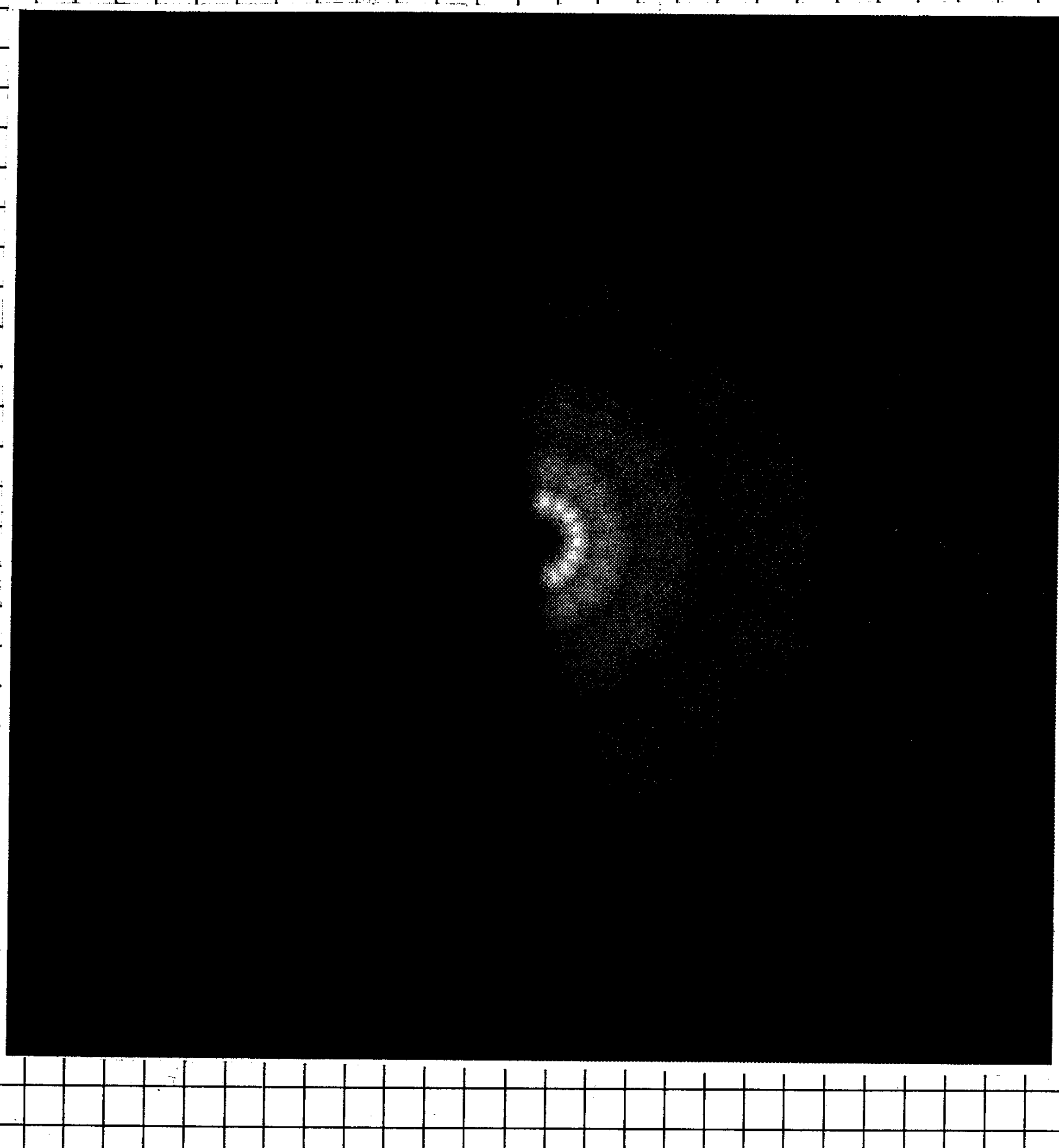
$v_0 = 9.6$ cpi

$R = 1.8$

$B = 1.0$

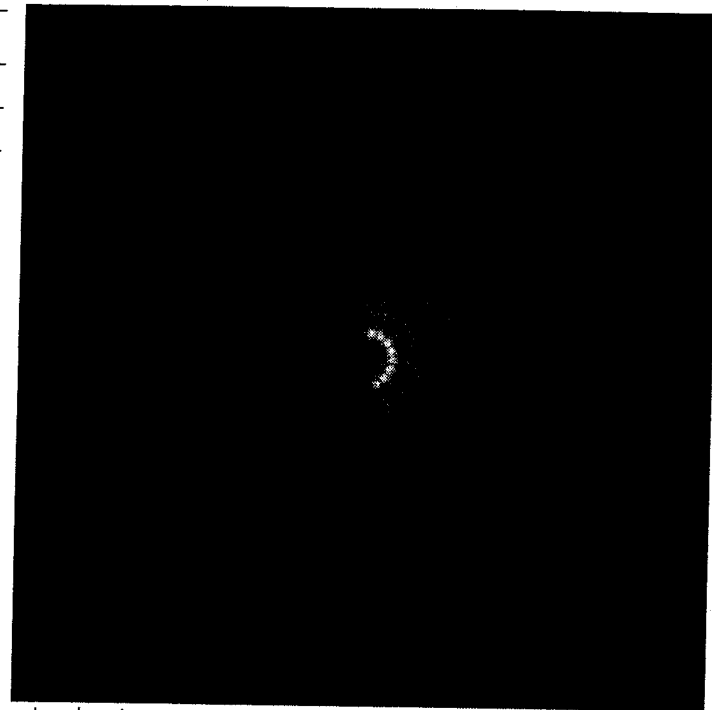
$n = 1/2$

Here it is: made by ph.



Another view made by klaros: 8/14/93

```
Filterbank:
NumRays = 8
NumFiltersPerRay = 5
TotNumFilters = 40
r0 = 0.037500
Common Ratio = 1.800000
Theta0 = -1.570796
Radial Spacing = 0.360268
Eta = 0.500000
OBW = 1.000000
gamma = 0.111111
```



```
Filter 0
Ray = 0
IndexOnRay = 0
Theta = -1.390662
r0 = 0.037500
u0 = 0.006719
v0 = -0.036893
sigma = 10.600415
GMax = 53.142597

Filter 1
Ray = 0
IndexOnRay = 1
Theta = -1.390662
r0 = 0.067500
u0 = 0.012093
v0 = -0.066408
sigma = 5.889119
GMax = 29.523666

Filter 2
Ray = 0
IndexOnRay = 2
Theta = -1.390662
r0 = 0.121500
u0 = 0.021768
v0 = -0.119534
sigma = 3.271733
GMax = 16.402037

Filter 3
Ray = 0
IndexOnRay = 3
Theta = -1.390662
r0 = 0.218700
u0 = 0.039183
v0 = -0.215161
sigma = 1.817630
GMax = 9.112243

Filter 4
Ray = 0
IndexOnRay = 4
Theta = -1.390662
r0 = 0.393660
u0 = 0.070529
v0 = -0.387290
sigma = 1.009794
GMax = 5.062357

Filter 5
Ray = 1
IndexOnRay = 0
Theta = -1.030395
r0 = 0.037500
u0 = 0.019293
v0 = -0.032156
sigma = 10.600415
GMax = 53.142597

Filter 6
Ray = 1
IndexOnRay = 1
Theta = -1.030395
r0 = 0.067500
u0 = 0.034727
v0 = -0.057881
sigma = 5.889119
GMax = 29.523666

Filter 7
Ray = 1
IndexOnRay = 2
Theta = -1.030395
r0 = 0.121500
u0 = 0.062509
v0 = -0.104186
sigma = 3.271733
GMax = 16.402037

Filter 8
Ray = 1
IndexOnRay = 3
Theta = -1.030395
r0 = 0.218700
u0 = 0.112517
v0 = -0.187536
sigma = 1.817630
GMax = 9.112243

Filter 9
Ray = 1
IndexOnRay = 4
Theta = -1.030395
r0 = 0.393660
u0 = 0.202530
v0 = -0.337564
sigma = 1.009794
GMax = 5.062357

Filter 10
Ray = 2
IndexOnRay = 0
Theta = -0.670127
r0 = 0.037500
u0 = 0.029390
v0 = -0.023291
sigma = 10.600415
GMax = 53.142597

Filter 11
Ray = 2
IndexOnRay = 1
Theta = -0.670127
r0 = 0.067500
u0 = 0.052903
v0 = -0.041923
sigma = 5.889119
GMax = 29.523666

Filter 12
Ray = 2
IndexOnRay = 2
Theta = -0.670127
r0 = 0.121500
u0 = 0.095225
v0 = -0.075462
sigma = 3.271733
GMax = 16.402037

Filter 13
Ray = 2
IndexOnRay = 3
Theta = -0.670127
r0 = 0.218700
u0 = 0.171405
v0 = -0.135831
sigma = 1.817630
GMax = 9.112243

Filter 14
Ray = 2
IndexOnRay = 4
Theta = -0.670127
r0 = 0.393660
u0 = 0.308528
v0 = -0.244496
sigma = 1.009794
GMax = 5.062357

Filter 15
Ray = 3
IndexOnRay = 0
Theta = -0.309859
r0 = 0.037500
u0 = 0.035714
v0 = -0.011435
sigma = 10.600415
GMax = 53.142597

Filter 16
Ray = 3
IndexOnRay = 1
Theta = -0.309859
r0 = 0.067500
u0 = 0.064285
v0 = -0.020582
sigma = 5.889119
GMax = 29.523666

Filter 17
Ray = 3
IndexOnRay = 2
Theta = -0.309859
r0 = 0.121500
u0 = 0.115714
v0 = -0.037048
sigma = 3.271733
GMax = 16.402037

Filter 18
Ray = 3
IndexOnRay = 3
Theta = -0.309859
r0 = 0.218700
u0 = 0.208285
v0 = -0.066687
sigma = 1.817630
GMax = 9.112243

Filter 19
Ray = 3
IndexOnRay = 4
Theta = -0.309859
r0 = 0.393660
u0 = 0.374913
v0 = -0.120037
sigma = 1.009794
GMax = 5.062357

Filter 20
Ray = 4
IndexOnRay = 0
Theta = 0.050409
r0 = 0.037500
u0 = 0.037452
v0 = 0.001890
sigma = 10.600415
GMax = 53.142597

Filter 21
Ray = 4
IndexOnRay = 1
Theta = 0.050409
r0 = 0.067500
u0 = 0.067414
v0 = 0.003401
sigma = 5.889119
GMax = 29.523666

Filter 22
Ray = 4
IndexOnRay = 2
Theta = 0.050409
r0 = 0.121500
u0 = 0.121346
v0 = 0.006122
sigma = 3.271733
GMax = 16.402037

Filter 23
Ray = 4
IndexOnRay = 3
Theta = 0.050409
r0 = 0.218700
u0 = 0.218422
v0 = 0.011020
sigma = 1.817630
GMax = 9.112243

Filter 24
Ray = 4
IndexOnRay = 4
Theta = 0.050409
r0 = 0.393660
u0 = 0.393160
v0 = 0.019835
sigma = 1.009794
GMax = 5.062357

Filter 25
Ray = 5
IndexOnRay = 0
Theta = 0.410677
r0 = 0.037500
u0 = 0.034382
v0 = 0.014971
sigma = 10.600415
GMax = 53.142597

Filter 26
Ray = 5
IndexOnRay = 1
Theta = 0.410677
r0 = 0.067500
u0 = 0.061887
v0 = 0.026948
sigma = 5.889119
GMax = 29.523666

Filter 27
Ray = 5
IndexOnRay = 2
Theta = 0.410677
r0 = 0.121500
u0 = 0.111397
v0 = 0.048506
sigma = 3.271733
GMax = 16.402037

Filter 28
Ray = 5
IndexOnRay = 3
Theta = 0.410677
r0 = 0.218700
u0 = 0.200515
v0 = 0.087312
sigma = 1.817630
GMax = 9.112243

Filter 29
Ray = 5
IndexOnRay = 4
Theta = 0.410677
r0 = 0.393660
u0 = 0.360928
v0 = 0.157161
sigma = 1.009794
GMax = 5.062357

Filter 30
Ray = 6
IndexOnRay = 0
Theta = 0.770944
r0 = 0.037500
u0 = 0.026897
v0 = 0.026130
sigma = 10.600415
GMax = 53.142597

Filter 31
Ray = 6
IndexOnRay = 1
Theta = 0.770944
r0 = 0.067500
u0 = 0.048415
v0 = 0.047035
sigma = 5.889119
GMax = 29.523666

Filter 32
Ray = 6
IndexOnRay = 2
Theta = 0.770944
r0 = 0.121500
u0 = 0.087146
v0 = 0.084663
sigma = 3.271733
GMax = 16.402037

Filter 33
Ray = 6
IndexOnRay = 3
Theta = 0.770944
r0 = 0.218700
u0 = 0.156863
v0 = 0.152393
sigma = 1.817630
GMax = 9.112243

Filter 34
Ray = 6
IndexOnRay = 4
Theta = 0.770944
r0 = 0.393660
u0 = 0.282354
v0 = 0.274307
sigma = 1.009794
GMax = 5.062357

Filter 35
Ray = 7
IndexOnRay = 0
Theta = 1.131212
r0 = 0.037500
u0 = 0.015959
v0 = 0.033935
sigma = 10.600415
GMax = 53.142597

Filter 36
Ray = 7
IndexOnRay = 1
Theta = 1.131212
r0 = 0.067500
u0 = 0.028726
v0 = 0.061083
sigma = 5.889119
GMax = 29.523666

Filter 37
Ray = 7
IndexOnRay = 2
Theta = 1.131212
r0 = 0.121500
u0 = 0.051706
v0 = 0.109949
sigma = 3.271733
GMax = 16.402037

Filter 38
Ray = 7
IndexOnRay = 3
Theta = 1.131212
r0 = 0.218700
u0 = 0.093071
v0 = 0.197908
sigma = 1.817630
GMax = 9.112243

Filter 39
Ray = 7
IndexOnRay = 4
Theta = 1.131212
r0 = 0.393660
u0 = 0.167527
v0 = 0.356234
sigma = 1.009794
GMax = 5.062357
```