On the Solution of Fiber Orientation in Two-Dimensional Homogeneous Flows

SAMEER AKBAR and M. CENGIZ ALTAN*

School of Aerospace and Mechanical Engineering
University of Oklahoma
Norman, Oklahoma 73019

This work presents an analytical technique to describe the orientation behavior of short fibers in arbitrary two-dimensional homogeneous flows. It is shown that the fiber orientation, specified by a unit vector, can be analytically calculated at any instant using any initial orientation and flow kinematics. The rotation of a fiber with the bulk fluid deformation is expressed in terms of orientation vector components by utilizing an equivalent strain tensor calculated from the fluid kinematics. This technique is then used to evaluate the orientation behavior of a large number of fibers starting from different initial orientations, representing an orientation state. The orientation distribution function is generated statistically by considering the frequency distribution curve of the orientation of the large number of fibers. It is shown that using a combination of analytical solutions and statistical methods provides a convenient description of fiber orientation behavior. The accuracy of the generated orientation distribution function is found to be dependent on the number of fibers used in the analytical solution. The statistical orientation distribution function is compared with the exact solutions for certain homogeneous flows and found to be in close agreement.

INTRODUCTION

In the processing of short fiber-reinforced materials, numerical simulation of fiber orientation has lately been the focus of numerous research endeavors. These materials generally comprise short fibers embedded in a polymeric matrix, which serves as the reinforcing agent. The properties of short fiber-reinforced materials are known to be anisotropic because of the fiber orientation induced by the flow kinematics during processing. Therefore, enhancement of material properties can be achieved if the final fiber orientation can be predicted accurately.

A number of researchers have worked towards modeling the fiber orientation in different kinds of flows by using both analytical and numerical techniques. Jeffery's equations for the motion of an ellipsoidal particle in viscous medium (1) form the basis of most of the research performed in this field. Okagawa, et al. (2, 3), Chaffey, et al. (4), and Goldsmith and Mason (5) have analyzed Jeffery's equations to predict fiber behavior in various homogeneous flows considering different particle shapes. In addition, Brenner (6) has also presented explicit analytical results which describe the behavior of particles of various geometries in homogeneous flows. To verify and validate the analytical results, experimental investigation has also been performed, studying the factors affecting fiber behavior (7, 8).

Givler (9) and Givler, et al. (10), were the first to provide a numerical scheme for the solution of Jeffery's orientation equations for non-homogeneous flows with spatially non-uniform velocity gradients. In Givler's work, using local velocity gradients, the planar orientation angle $\phi$ is solved along the streamlines with a finite element technique. In essence, the proposed solution technique can be viewed as the numerical solution of the differential equation that governs the time rate of change of the orientation angle $\phi$. Within this framework, it has been common practice to express the orientation equations in terms of the time rate of change of the orientation angles $\phi$, or $\phi$ and $\theta$ for two- or three-dimensional cases, respectively.

For two-dimensional flows and planar orientations, the time rate of change of the orientation angle $\phi$, measured from $X_1$ axis can be expressed...
as
\[
\frac{d\phi}{dt} = \frac{1}{2} \left[ \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right] + \frac{\lambda}{2} \left[ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right] \cos 2\phi
\]
\[
+ \left[ \frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_1} \right] \sin 2\phi \tag{1}
\]

Similarly, for three-dimensional flows and orientations, the time rates of change of the orientation angles \(\phi\) and \(\theta\) are given by
\[
\frac{d\phi}{dt} = \frac{1}{2} \left[ \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right] + \frac{\lambda}{2} \left[ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right] \cos 2\phi
\]
\[
+ \left[ \frac{\partial u_3}{\partial x_2} - \frac{\partial u_4}{\partial x_1} \right] \sin 2\phi
\]
\[
+ \lambda \left[ \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1} \right] \cos \phi \cot \theta
\]
\[
+ \frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_1} \right] \sin \phi \cot \theta
\]
\[
+ \frac{\partial u_3}{\partial x_3} - \frac{\partial u_4}{\partial x_1} \sin 2\phi \tag{2}
\]

In summary, in all the previous works dealing with fiber orientation in complex flows, the orientation equations are solved numerically. The proposed numerical methods either directly utilize the numerically obtained velocity gradients on the nodal points (i.e., Eulerian formulation), or follow the particles along the streamlines and use local velocity gradients (i.e., Lagrangian formulation). Although the velocity gradients along the streamlines are spatially non-uniform, the local gradients interpolated from the surrounding nodal points can be employed as local constants for the numerical solutions. In this paper, we present the analytical solutions of Jeffery’s equation for an ellipsoid of revolution in arbitrary two-dimensional homogeneous (i.e., spatially uniform velocity gradients) flows. Once the analytical solutions are known, it is straightforward to evaluate the orientation behavior of a single as well as a large number of fibers. Hence, for complex flows, the analytical solution of orientation distribution along the flow streamlines can be accomplished with relative ease by considering local velocity gradients and tracing numerous fibers each starting from specified initial orientations. It should be noted that the specification of initial (inlet) orientation for each fiber can be selected to represent a variety of orientation states such as randomly oriented, fully or partially aligned fibers.
are compared with existing analytical solutions, demonstrating the validity of this method.

**ORIENTATION BEHAVIOR OF A FIBER**

This section gives the analytical solution of the orientation equation of a fiber suspended in a known velocity field, starting from a given initial orientation. The fiber orientation is specified by a unit vector and orientation change is evaluated by considering the time rate of change of the unit vector. A discussion on the validity of using Jeffery’s equation for fiber suspensions without considering the effect of fibers on the bulk flow kinematics is also included. Later, example solutions are obtained for simple shear, planar elongation, shearing-stretching, and rotating-stretching flows.

**Equations and Definitions**

For any two-dimensional homogeneous flow field, the velocity gradient tensor can be specified as,

$$u_{ij} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} c & c_1 \\ c_2 & -c \end{pmatrix}$$

where $c$, $c_1$, and $c_2$ are arbitrary constants and the trace of $u_{ij}$ should be zero to satisfy the conservation of mass for incompressible flow. This velocity gradient tensor can be used to represent different kinds of homogeneous flows depending on the values of the velocity gradients.

The equation for time rate of change of the unit orientation vector, effectively represents the rotation of a fiber. The unit vector, constrained by its inextensibility, cannot affinely follow the bulk fluid deformation. Hence, the stretching part of the affine motion is subtracted from the motion of the fluid element which coincides with the unit vector. For finite aspect ratio fibers, the equation of fiber rotation can be expressed in the following form:

$$\dot{p}_i = \left( \Omega_{ij} + \lambda D_{ij} \right) p_j - \lambda p_i p_k D_{ik}$$

where $\Omega_{ik}$ and $D_{ik}$ are the vorticity and deformation rate tensors, respectively.

$$\Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

$$D_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

In the above equations, $i, j = 1, 2$, and summation over repeated indices is implied throughout the text. For infinite aspect ratio fibers $\alpha_p \rightarrow \infty$, hence, $\lambda = 1$. Equation 6 is also known as Jeffery’s equation of fiber (i.e. ellipsoid of revolution) orientation.

Jeffery’s equation is derived considering a single ellipsoidal particle and is not suitable for the representation of the rheology of non-dilute fiber suspensions where the fibers are hydrodynamically coupled. The assumptions existing in Jeffery’s analysis precludes its direct use if the presence of fibers in the suspension affects the bulk flow kinematics. The assumptions involved in the Jeffery’s equation have been reviewed in numerous publications; however, it is important to emphasize two principal ones regarding the length scales of the flow and suspended particles:

1. The undisturbed flow is steady with spatially uniform velocity gradients over a length scale much larger than characteristic particle size. Hence, the length scale of the disturbance produced in the neighborhood of the particle is much smaller than the length scale of bulk motion.
2. Reynolds number based on particle size is much smaller than unity indicating either very slow flow or very small particles.

The usefulness of using Jeffery’s equation alone for the fiber suspensions is addressed by several researchers. It is argued that the fibers affect the bulk flow kinematics at very low volume concentrations which necessitates the use of an anisotropic constitutive equation for the rheological characterization of the suspension. However, provided that the particle sizes are small compared to overall deformation scales, one can show that there exists a volume fraction low enough to validate the use of Jeffery’s equation. In other words, if the stress contributions from the presence of fibers (which are shown to be linearly dependent on the fiber volume fraction at low fiber concentrations) are much less than the Newtonian viscous stresses of the flow field, the orientation field can be accurately described by the Jeffery’s analysis.

**Analytical Solution Technique for the Fiber Orientation**

The solution for Eq 6, with the initial condition $\dot{p} = p^s$, has been shown in (16) as,

$$p_i = \frac{E_{11} p_1^s}{\left( E_{11} E_{11} E_{11} p_1^s p_1^s \right)^{1/2}}$$

where $E_{ij}$’s are components of a strain tensor defined in Eq 10 below. The numerator of Eq 9 is easily seen to be the vector which would result if $p^s$ deformed affinely with the bulk fluid deformation between times $t$ and $t^*$. Therefore, the denominator is the magnitude of this deformation and the division normalizes $\dot{p}$ to a unit vector in the deformed state.

For particles with infinite aspect ratio, $E_{ij}$ represents the strain tensor components based on the bulk deformation and is defined as,

$$E_{ij} = \frac{\partial x_i}{\partial x_j}$$

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where $x_i$ and $x_j$ are the coordinates of a fluid particle at times $t$ and $t'$, respectively. Therefore, the orientation evolution of infinite aspect ratio fibers can be analytically calculated from Eq 9 if $E_{ij}$ components are determined in terms of velocity gradients. For finite aspect ratio fibers, Eq 10 is modified, and the rate of change in strain tensor is given as (15).

$$\frac{dE_{ij}}{dt} = (\Omega_{ik} + \lambda D_{ik})E_{kj}$$  \hspace{1cm} (11)

Naturally, for infinite aspect ratio (i.e. $\lambda = 1$), Eq 11 reduces to Eq 10.

For a planar orientation description, the initial orientation angle $\phi_0$, is related to the initial orientation vector $p^0$ as,

$$p_1^0 = \cos \phi_0$$

$$p_2^0 = \sin \phi_0$$  \hspace{1cm} (12)

where $p_1^0$ and $p_2^0$ are the components of $p^0$ along the $X_1$ and $X_2$ axes, respectively. Similarly, at any instant, the orientation vector components are related to the orientation angle $\phi$ by,

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix}$$  \hspace{1cm} (13)

where, $p_1$ and $p_2$ are the components of the orientation vector. Therefore, the orientation angle can be calculated from,

$$\tan \phi = \frac{p_2}{p_1}$$  \hspace{1cm} (14)

where the angle $\phi$ is measured from $X_1$. Obviously, if the fluid kinematics and initial orientation are known, the orientation angle can be calculated by using Eqs 9 and 14.

To calculate the strain tensor in terms of velocity gradients, four first-order coupled differential equations obtained from Eq 11 have to be solved with the initial condition $E = I$ (unit tensor).

$$\frac{dE_{11}}{dt} = (\Omega_{11} + \lambda D_{11})E_{11} + (\Omega_{12} + \lambda D_{12})E_{21}$$

$$\frac{dE_{12}}{dt} = (\Omega_{11} + \lambda D_{11})E_{12} + (\Omega_{12} + \lambda D_{12})E_{22}$$

$$\frac{dE_{21}}{dt} = (\Omega_{21} + \lambda D_{21})E_{11} + (\Omega_{22} + \lambda D_{22})E_{21}$$

$$\frac{dE_{22}}{dt} = (\Omega_{21} + \lambda D_{21})E_{12} + (\Omega_{22} + \lambda D_{22})E_{22}$$  \hspace{1cm} (15)

where Eqs 4, 7, and 8 are used for $\lambda$, $\Omega_{ij}$, and $D_{ij}$, respectively. It is found that $w^2$, shown below, appears as a coefficient which governs the solution families of Eq 15.

$$w^2 = \frac{\lambda^2 (B^2 + 4c^2) - A^2}{4}$$  \hspace{1cm} (16)

where, $B = c_1 + c_2$ and $A = c_1 - c_2$. Depending on the value of $w^2$, three different categories of solutions are possible.

When $w^2 = 0$,

$$E_{ij} = \begin{bmatrix} c\lambda t + 1 \\ (\lambda B - A)t \\ 2 \end{bmatrix}$$  \hspace{1cm} (17)

When $w^2 > 0$,

$$E_{ij} = \begin{bmatrix} \frac{w \cos h(\omega t) + c\lambda \sin h(\omega t)}{\omega} \\ -\frac{(A - \lambda B)\sin h(\omega t)}{2w} \\ \frac{(A + \lambda B)\sin h(\omega t)}{w} \end{bmatrix}$$  \hspace{1cm} (18)

$$E_{ij} = \begin{bmatrix} \frac{w \cos (\omega t) + c\lambda \sin (\omega t)}{\omega} \\ -\frac{\omega \cos h(\omega t) - c\lambda \sin h(\omega t)}{2w} \\ \frac{(\lambda B + A)\sinh(\omega t)}{w} \end{bmatrix}$$  \hspace{1cm} (19)

where in all three cases $w$ is defined as $w = \sqrt{|w^2|}$. Equations 17, 18, and 19 are the possible solutions for the strain tensor and can be used in Eq 9 to find the components of the orientation vector. Although the above expressions for $E_{ij}$ can be used for three-dimensional orientation fields without further modifications, in this paper we limit our discussion to planar orientations. For infinite aspect ratio fibers where $\lambda = 1$, the three different $E_{ij}$'s can be viewed as the strain tensors of simple shear, elongation, and rotation dominant flows, respectively (13).
To calculate \( p_1 \) and \( p_2 \), Eq 9 can be written explicitly as,

\[
p_1 = \frac{p_{*1} E_{11} + p_{*2} E_{12}}{X^{1/2}}
\]

\[
p_2 = \frac{p_{*2} E_{22} + p_{*1} E_{21}}{X^{1/2}}
\]

\[
X = (E_{11}^2 + E_{21}^2) p_{*1}^2 + (E_{11} E_{12} + E_{21} E_{22}) 2 p_{*1} p_{*2} + (E_{12}^2 + E_{22}^2) p_{*2}^2
\]

Depending on the value of \( w^2 \), Eqs 17, 18, or 19 can be used in Eq 20 to calculate the orientation vector components.

The method presented above can also be applied to three-dimensional homogeneous flows. Equation 11 can be directly used with the three-dimensional vorticity and strain rate tensors. The procedure is straightforward but tedious. Nine coupled differential equations need to be solved to obtain the three-dimensional \( E_{ij} \) components. Then Eq 9 can be employed without any modification to obtain the final orientation of a fiber subjected to any three-dimensional flow field with spatially constant velocity gradients.

**Example Cases**

To illustrate the use and validity of these analytical results, a few example flows are investigated and the results are compared with existing solutions. The initial orientation in most cases was taken as \( \pi/2 \), unless otherwise stated, and fiber motion was analyzed between \( -\pi/2 \) and \( +\pi/2 \).

**Simple Shear Flow**

For simple shear flow, the velocity gradient tensor is given by,

\[
\begin{pmatrix}
  0 & c_1 \\
  0 & 0
\end{pmatrix}
\]

From Eqs 16 and 21, it is found that \( w^2 \lesssim 0 \), \( A = c_1 \), and \( B = c_1 \). Therefore, Eq 19 gives the following strain tensor:

\[
E_{ij} = \begin{pmatrix}
  \cos(wt) & \frac{(\lambda + 1)c_1 \sin(wt)}{2w} \\
  \frac{(\lambda - 1)c_1 \sin(wt)}{2w} & \cos(wt)
\end{pmatrix}
\]

where,

\[
w = \left( \sqrt{\lambda^2 - 1} \right) \frac{\dot{\gamma}}{2}
\]

and shear rate \( \dot{\gamma} = c_1 \).

The orientation angle at any instant can be calculated from,

\[
\tan \phi = a_p \tan \left( \frac{2\pi}{\gamma} \left( a_p + a_p^{-1} \right) + K \right)
\]

\[
K = \tan^{-1} \left( \frac{\tan \phi_0}{a_p} \right)
\]

where \( K \) is a function of the initial orientation \( \phi_0 \). From the above expression it is apparent that the orientation angle is a function of shear, aspect ratio, and time. The fiber rotates with a time period given by,

\[
T = \frac{2\pi}{\gamma} \left( a_p + a_p^{-1} \right)
\]

**Figure 1** shows a plot of orientation change with total shear \( \dot{\gamma} t \) for simple shear flow. To illustrate the effect of aspect ratio on orientation behavior, three different aspect ratios are considered. As can be predicted from Eq 25, the fibers with small aspect ratio have a shorter time period, hence tumble rapidly after every half period. For large aspect ratio fibers, the time period is longer, showing that the fibers have greater tendency to remain aligned in the flow direction, and higher total shear is required to observe tumbling, as can be seen for \( a_p = 50 \). Similar results have been shown earlier by numerous researchers including Jeffery (1), Okagawa, et al. (3), and Brenner (6).

**Planar Elongational Flow**

For planar elongational flow, the velocity gradient tensor is given by,

\[
\begin{pmatrix}
  c & 0 \\
  0 & -c
\end{pmatrix}
\]

where \( c \) is the elongation rate. From Eqs 16 and 26 we find that \( w^2 > 0 \), \( A = 0 \), and \( B = 0 \). Therefore,
from Eq 18, the strain tensor becomes,

\[ E_{ij} = \begin{pmatrix} \sin h(\varepsilon \lambda) + \cos h(\varepsilon \lambda) & 0 \\ 0 & \cos h(\varepsilon \lambda) - \sin h(\varepsilon \lambda) \end{pmatrix} \]

(27)

where \( \varepsilon = ct \) is total elongation.

Utilizing the strain tensor obtained in Eq 27, the orientation angle at any instant is found to be,

\[ \tan \phi = \left( \frac{1 - \tan h(\lambda \varepsilon)}{1 + \tan h(\lambda \varepsilon)} \right) \tan \phi_0 \]

(28)

Figure 2 shows a plot of orientation change with total elongation for planar elongation flow. It can be seen that starting from an orientation less than \( \pi / 2 \) (89.9 degrees in Fig. 2), the fiber aligns rapidly with the elongation direction, which in this case is along \( \phi = 0 \). If the initial orientation is \( \pi / 2 \), it is found that any perturbation causes the fiber to rapidly align with the elongation direction. Therefore, \( \phi = 0 \) is the stable equilibrium position whereas \( \phi = \pi / 2 \) is the unstable equilibrium.

**Shearing-Stretching Flow**

In shearing-stretching flow, the velocity gradient tensor is expressed as a combination of simple shear and planar elongation flow components. Therefore, the velocity gradient tensor can be written as,

\[ u_{ij} = \begin{pmatrix} -c & c_1 \\ 0 & c \end{pmatrix} = \begin{pmatrix} -1 & c_1 / c \\ 0 & 1 \end{pmatrix} \]

(29)

The signs of the velocity gradients have been chosen such that the shear and elongation components compete against each other, trying to align the fibers along different directions. The relative magnitude of \( c_1 \), with respect to \( c \) dictates the dominating flow type and the orientation behavior. If \( c \rightarrow \infty \), the flow is purely elongational whereas if \( c_1 \rightarrow \infty \) the flow reduces to simple shear. In the case considered, \( A = B = c_1 / c \), therefore from Eq 16,

\[ w^2 = \frac{c_1^2}{c^2} (\lambda^2 - 1) + 4 \lambda^2 \]

(30)

The magnitude of \( w^2 \) can be equal to, less than, or greater than zero depending on the value of the numerator in Eq 30. If,

\[ \frac{c_1^2}{c^2} < \frac{4 \lambda^2}{1 - \lambda^2} \]

then \( w^2 = 0 \) and Eq 17 is utilized to represent the strain tensor. If,

\[ \frac{c_1^2}{c^2} > \frac{4 \lambda^2}{1 - \lambda^2} \]

then \( w^2 > 0 \) and Eq 18 will represent the strain tensor. This is the case for \( c_1 / c = 0.2 \) and \( \alpha_p = 5.0 \) as shown in Fig. 3a. The strain tensor is found to be,
and, the orientation behavior can be expressed as
\[
\tan \phi = \frac{\frac{c_1}{c} (\lambda - 1) \tan h(\omega t)}{2 \tan \phi_s} + w + \lambda \tan h(\omega t)
\]
\[
\tan \phi = \frac{\frac{c_1}{c} (\lambda - 1) \tan h(\omega t)}{2 \tan \phi_s} + w + \lambda \tan h(\omega t)
\]
\[
\frac{c_1}{c}(\lambda + 1) \tan h(wt) \quad w - \lambda \tan h(wt)
\]
\[
\tan \phi = \frac{\frac{c_1}{c} (\lambda + 1) \tan h(wt)}{2 \tan \phi_s} + w + \lambda \tan h(\omega t)
\]
\[
\tan \phi = \frac{\frac{c_1}{c} (\lambda + 1) \tan h(wt)}{2 \tan \phi_s} + w + \lambda \tan h(\omega t)
\]
\[
\tan \phi = \frac{\frac{c_1}{c} (\lambda + 1) \tan h(wt)}{2 \tan \phi_s} + w + \lambda \tan h(\omega t)
\]

In the equation above, as \( t \to \infty \), the hyperbolic term approaches unity and the expression for the steady state orientation reduces to,
\[
\tan \phi = \frac{\frac{c_1}{c} (\lambda + 1) \tan h(wt)}{2 \tan \phi_s} + w + \lambda \tan h(\omega t)
\]
\[
\tan \phi = \frac{\frac{c_1}{c} (\lambda + 1) \tan h(wt)}{2 \tan \phi_s} + w + \lambda \tan h(\omega t)
\]
\[
\tan \phi = \frac{\frac{c_1}{c} (\lambda + 1) \tan h(wt)}{2 \tan \phi_s} + w + \lambda \tan h(\omega t)
\]

In Fig. 3a the effect of initial orientation angle is observed to be negligible and a stable equilibrium orientation of 84° is quickly attained by the fibers which can also be predicted from Eq 35.

Consider the case when
\[
\frac{c_1^2}{c^2} > \frac{4 \lambda^2}{1 - \lambda^2}
\]
\[
\frac{c_1^2}{c^2} > \frac{4 \lambda^2}{1 - \lambda^2}
\]

Here, \( w^2 < 0 \) and Eq 19 will be used for strain tensor. Such a case is shown in Fig. 3a with \( c_1/c = 5.0 \) and \( \alpha_p = 5.0 \). From Eq 19 the strain tensor is given by
\[
E_{ij} = \begin{pmatrix}
w \cos(wt) - \lambda \sin(wt) & \frac{c_1}{c}(\lambda + 1) \sin(wt) \\
\frac{c_1}{c}(\lambda - 1) \sin(wt) & w \cos(wt) + \lambda \sin(wt)
\end{pmatrix}
\]
\[
E_{ij} = \begin{pmatrix}
w \cos(wt) - \lambda \sin(wt) & \frac{c_1}{c}(\lambda + 1) \sin(wt) \\
\frac{c_1}{c}(\lambda - 1) \sin(wt) & w \cos(wt) + \lambda \sin(wt)
\end{pmatrix}
\]

and the fiber orientation is expressed as
\[
\tan \phi = \frac{\frac{c_1}{c} (\lambda - 1) \tan(\omega t)}{2 \tan \phi_s} + w + \lambda \tan(\omega t)
\]
\[
\tan \phi = \frac{\frac{c_1}{c} (\lambda - 1) \tan(\omega t)}{2 \tan \phi_s} + w + \lambda \tan(\omega t)
\]

It is apparent from Eq 38 that the orientation change is periodic. The time period of fiber rotation is found to be a function of \( c_1/c \) and aspect ratio,
\[
T = \frac{4 \pi}{\sqrt{\left(\frac{c_1}{c}\right)^2 (\lambda^2 - 1) + 4 \lambda^2}}
\]
\[
T = \frac{4 \pi}{\sqrt{\left(\frac{c_1}{c}\right)^2 (\lambda^2 - 1) + 4 \lambda^2}}
\]

The plot for \( c_1/c = 5.0 \) in Fig. 3a depicts the time period to be independent of the initial orientation and all fibers tumble rapidly after every half period in the same way as they do in simple shear flow. For the given flow field, the time period is calculated to be 23.3 s from Eq 39.

Figure 3b shows a plot for the same flow field as in Fig. 3a, but with \( \alpha_p = 10.0 \). It is found that for \( c_1/c = 0.2 \) and 5.0, \( w^2 > 0 \). Therefore, no tumbling is observed and from Eq 35 the equilibrium orientation angles are calculated to be 84.2° and 20.2°, respectively.

The important observation from Figs. 3a and 3b is the equilibrium or final orientation attained by the fibers. For shear dominant flows, i.e. \( c_1 > c \), the equilibrium orientation is closer to the steady state orientation for simple shear whereas for elongation dominant flows, i.e. \( c > c_1 \), it is closer to the steady state orientation for planar elongation flow. For example, as \( t \to \infty \), for \( c_1/c = 1.2 \), 2.0, and 5.0, we obtain \( \phi = 58°, 46.17°, \) and 21°, respectively, while for \( c_1/c = 0.8 \), 0.5, and 0.1, \( \phi = 67.4°, 75.4°, \) and 87°.

Figures 3a and 3b also depict the effect of aspect ratio on fiber dynamics. Although the flow kinematics remain the same, it is seen that the aspect ratio of the fiber substantially affects its behavior. For \( \alpha_p = 5 \), the fiber is observed to tumble, while for \( \alpha_p = 10 \), the fiber rapidly attains a steady orientation.

Rotating-Stretching Flow

As the name implies, this kind of flow has rotational as well as stretching components. Hence, both rotational and planar elongational flow components are present in the velocity gradient tensor. Therefore, for such flows the velocity gradient tensor can be expressed as
\[
\mathbf{u}_{ij} = \begin{pmatrix}
c & c_1 \\
c_1 & -c_1/c
\end{pmatrix} = \begin{pmatrix}
1 & c_1/c \\
-c_1/c & -1
\end{pmatrix}
\]
\[
\mathbf{u}_{ij} = \begin{pmatrix}
c & c_1 \\
c_1 & -c_1/c
\end{pmatrix} = \begin{pmatrix}
1 & c_1/c \\
-c_1/c & -1
\end{pmatrix}
\]

The relative magnitude of \( c_1 \) with respect to \( c \) determines the dominant flow type. If \( c \to \infty \), the flow is
completely elongational whereas if \( c_1 \to \infty \), the flow is purely rotational. In the present case, \( A = 2(c_1/c) \) and \( B = 0 \). Therefore, from Eq. 16,

\[
\omega^2 = \lambda^2 - \frac{c_1^2}{c^2} \tag{41}
\]

Once again, \( \omega^2 \) can be equal to, less than, or greater than zero depending on the value of the above expression. For the case when \( \omega^2 > 0 \), the strain tensor is given by Eq. 18 as

\[
E_{ij} = \begin{pmatrix}
\frac{w \cos h(\omega t) + \lambda \sin h(\omega t)}{w} & \frac{c_1 \sin h(\omega t)}{w} \\
\frac{-\frac{c_1}{c} \sin h(\omega t)}{w} & \frac{w \cos h(\omega t) - \lambda \sin h(\omega t)}{w}
\end{pmatrix}
\tag{42}
\]

The final orientation can be calculated from

\[
\tan \phi = \frac{\frac{-c_1}{c} \tan h(\omega t)}{\tan \phi_*} + \omega - \lambda \tan h(\omega t)
\]

\[
\tan \phi = \frac{w + \lambda \tan h(\omega t) + c_1 \tan h(\omega t)}{\tan \phi_*} + \omega - \lambda \tan h(\omega t) \tag{43}
\]

As \( t \to \infty \), the hyperbolic term approaches unity and the above expression reduces to

\[
\tan \phi = \frac{-c_1}{c} \tan h(\omega t) + \omega - \lambda \tan h(\omega t) \tag{44}
\]

Figures 4a and 4b are plots showing the variation in orientation angle as a function of total shear for rotating-stretching flow.

From Eq. 41 it is found that for \( c_1/c = 0.2 \) and \( \alpha_p = 5.0 \), \( \omega^2 > 0 \). It can be predicted from Eq. 44 that as \( t \to \infty \), the fiber attains an equilibrium position at \( \phi = -6.25^\circ \). The same kind of behavior is observed in Fig. 4b for \( c_1/c = 0.2 \) and \( \alpha_p \to \infty \), where the equilibrium position is at \( \phi = -5.76^\circ \).

For \( c_1/c = 3 \) and 5, \( \omega^2 < 0 \) and Eq. 19 gives the strain tensor

\[
E_{ij} = \begin{pmatrix}
\frac{w \cos [\omega t] + \lambda \sin [\omega t]}{w} & \frac{-c_1}{c} \sin [\omega t] \\
\frac{-\frac{c_1}{c} \sin [\omega t]}{w} & \frac{w \cos [\omega t] - \lambda \sin [\omega t]}{w}
\end{pmatrix}
\tag{45}
\]

Therefore, orientation at any instant can be expressed as

\[
\tan \phi = \frac{-c_1}{c} \tan [\omega t] + \omega - \lambda \tan [\omega t] \tag{46}
\]

Fig. 4a. Effect of \( c_1/c \) on the orientation behavior of fibers in rotating-stretching flows; \( \alpha_p = 5.0 \).

Fig. 4b. Effect of \( c_1/c \) on the orientation behavior of fibers in rotating-stretching flows; \( \alpha_p \to \infty \).
Hence, the orientation change is periodic with the following time period:

\[ T = \frac{2\pi}{\sqrt{\lambda^2 - \frac{c^4}{c^2}}} \]  

(47)

Plots for \( c_1/c = 1, 3, \) and 5 in Fig. 4a, and \( c_1/c = 3 \) and 5 in Fig. 4b show that the fiber tumbles rapidly after every half period. It is seen that the effect of aspect ratio is not very pronounced for the rotation dominant flows.

For \( c_1/c = 1 \) and \( \lambda = 1 \) in Fig. 4b, \( w^2 = 0. \) Therefore, from Eq 17 the strain tensor becomes

\[ E_{ij} = \left[ \begin{array}{cc} t + 1 & t \\ t & 1 - t \end{array} \right] \]  

(48)

and the orientation at any instant can be given by

\[ \tan \phi = \frac{t + 1}{\tan \phi_o + 1 - t} \]  

(49)

As can be calculated from the expression above, starting from an initial orientation of \( \pi/2 \) the fiber quickly approaches an orientation of \( \pi/4. \) It can also be observed from Figs. 4a and 4b that if the strength of rotational and stretching components is same, tumbling takes place just as in simple shear flow. In fact, it can be proved that this flow is identical to simple shear flow if the \( X, X, \) axes are transformed along the principal flow direction and further rotated by \( \pi/4. \)

In order to describe a meaningful orientation state, the orientation behavior of large number of fibers, each starting from a specified initial orientation, can be evaluated independently using the derived analytical solutions. Once the orientations of large number of fibers at any instant are known, statistical analysis can be used to generate the orientation distribution function. A simple frequency distribution curve obtained from this data gives the orientation distribution function. It is logical to assume that accounting for higher number of fibers should give a closer approximation to the exact analytical orientation distribution function. A relative error criteria is defined based on the standard deviation of the analytical and statistical results. It is expected that the relative error will decrease as the number of fibers utilized is increased.

**STATISTICAL ORIENTATION DISTRIBUTION FUNCTION**

The orientation distribution function \( \psi(\hat{p}, t) \) gives the probability of having a fiber at any given orientation \( \hat{p} \) at a certain time \( t. \) Equivalently, in terms of orientation angles it is defined at the probability of finding a fiber between angles \( \phi_1 \) and \( \phi_2 \) at any instant.

In this section, instead of a single fiber, analytical solutions are applied to large number of fibers starting from specified initial orientations. Therefore, the orientation distribution of numerous fibers at any instant is known depending on the velocity field. Subsequently, the orientation distribution function is generated by using statistical analysis based on this orientation data. The orientation distribution function obtained by this method is compared with the existing analytical solutions and the comparisons are presented in terms of error in standard deviation between the analytical and statistical solutions.

**Analytical Calculation of Orientation Distribution Function**

For two-dimensional (planar) description of fiber orientation, one end of the fiber should not be distinguishable from the other. Therefore, the distribution function has a period \( \pi \), or

\[ \psi(\phi + \pi) = \psi(\phi) \]  

(50)

Also, it satisfies the normalization condition

\[ \int_{0}^{\pi} \psi(\phi) \, d\phi = 1 \]  

(51)

For homogeneous flows with negligible Brownian motion, the governing equation for distribution function becomes

\[ \frac{\partial \psi(\hat{p}, t)}{\partial t} = - \frac{\partial}{\partial p_i} \left[ \hat{p}_i \psi(\hat{p}, t) \right] \]  

(52)

where \( i = 1, 2 \) and \( \hat{p}_i \) is given by Eq 6. Equation 52 is a form of the Fokker-Planck equation and obtained by invoking the conservation principles in orientation space. The analytical solution to this equation is found to be

\[ \psi(\hat{p}, t) = \frac{1}{\pi} \left( \Delta_{ij} \psi(\hat{p}, t) \right)^{-1} \]  

(53)

where \( \Delta_{ij} \) is the inverse of the strain tensor \( E_{ij}, \) and is called the deformation gradient tensor.

\[ \Delta_{ij} = \frac{\partial x_i^*}{\partial x_j} \]  

(54)

Equation 53 is valid when the fibers are initially in random orientation. Applying the normalization condition given in Eq 51, the random initial condition is written as

\[ \psi(\phi, t = 0) = \frac{1}{\pi} \]  

(55)

Using these equations, the analytical distribution function expressions for various two- and three-dimensional flow fields have been shown in (13) for infinite aspect ratio fibers.
For simple shear flow, Eqs 13, 22, and 53 are used to obtain the expression for distribution function for finite aspect ratio particles as

$$
\psi(\tilde{p},t) = \frac{1}{\pi} \left\{ \cos^2 wt - \frac{c_1 \lambda \sin(2wt)\sin(2\phi)}{2w} 
+ \frac{c_1^2 \sin^2(wt)[\lambda^2 + 1 - 2\lambda \cos 2\phi]}{4w^2} \right\}^{-1}
$$

(56)

Figure 5 shows the orientation distribution function for $\alpha_p = 5, 10, \infty$. For the magnitudes of total shear considered, the preferred angle of orientation is not affected significantly by aspect ratio. However, the aspect ratio affects the degree of alignment, indicating that for a given total shear, the tendency of fibers to align in the preferred orientation is increased for higher aspect ratio particles.

Figure 6 depicts plots for higher total shear values. It can be seen that for high total shear, the preferred angle of orientation is approaching to the flow direction with increasing degree of alignment. Moreover, the orientation distribution function is shown to be substantially affected by the fiber aspect ratio. For high fiber aspect ratios, the time period for tumbling is longer and a higher degree of alignment can be obtained before tumbling (for infinite aspect ratio there is no tumbling). On the other hand, for low fiber aspect ratios, as fibers become aligned with the flow direction, a significant proportion of the fibers tumble. Thus, only a limited degree of alignment is observed.

In case of planar elongational flow, the expression for orientation distribution function for finite aspect ratio particles is obtained from Eqs 13, 27, and 53 as

$$
\psi(\tilde{p},t) = \frac{1}{\pi} \left\{ \cos h^2(\epsilon \lambda) + \sin h^2(\epsilon \lambda) 
- \sin h(2\epsilon \lambda)\cos(2\phi) \right\}^{-1}
$$

(57)

Figure 7 shows a plot of the orientation distribution function for $\epsilon_p = 5, 10, \infty$, generated from the above expression. In this case, the preferred angle of orientation is not affected by aspect ratio, and higher degree of alignment is achieved for higher fiber aspect ratios as shown in Fig. 7.

The analytical solution of Fokker-Planck equation as given in Eqs 56–57 is not available for nonrandom initial orientations. This severely restricts the applicability of distribution function for complex flows where the solution of orientation distribution (i.e., orientation state) along a streamline is needed. In order to overcome this difficulty, we propose a statistical method which makes it possible to generate the orientation distribution function by considering large numbers of fibers without solving the Fokker-Planck equation. Hence, the solution of orientation fields along streamlines of complex flows will be feasible by considering numerous fibers as the analytical solutions for individual fibers are readily available. However, the accuracy of the orientation distribution function generated by this method depends on the number of fibers utilized and will be investigated further.
Statistical Calculation of Orientation Distribution Function

The definition of orientation distribution function gives an indication that it can be recovered statistically if the orientations of large number of fibers are known. In this case, once the angles $\phi_1, \phi_2, \ldots, \phi_N$ are known for the $N$ number of fibers considered, plotting the normalized distribution function simply amounts to drawing the frequency distribution curve of the sample. In addition, it has to be normalized to satisfy the condition given by Eq 51. The number of angular intervals has to be chosen such that the error between analytical and statistical orientation distribution function is minimum. Since the error between analytical and statistical solutions is a direct function of the number of fibers, the acceptable number of fibers to be used can be easily determined by comparing the two solutions and defining an acceptable error value.

The relative error $e$ in standard deviation is defined as

$$e = \frac{\text{std. deviation}(\text{anal.}) - \text{std. deviation}(\text{stat.})}{\text{std. deviation}(\text{anal.})}$$

where the absolute value of the numerator is used. Justification for the use of this parameter lies in the fact that each distribution function pair being compared is normalized to satisfy Eq 51.

Simple Shear Flow

Figures 8a, 8b, and 8c are results of statistical analysis on the orientations of a large number of fibers. Figure 8a shows the histogram of actual orientation distribution at total shear $\gamma = 2$ by considering 18000 infinite aspect ratio fibers starting from random orientation. The number of fibers in each of the 180 bars of the histogram should add up to 18000. Figure 8b shows three curves for the normalized orientation distribution function. Two are drawn statistically by considering 360 and 1800 fibers while the third is the analytical solution from Eq 56. For purpose of clarity angular intervals of two degrees are considered in drawing the statistical curves as compared to one degree in Fig. 8a. It is seen that as the number of fibers are increased, the statistical results approximate the analytical results better and the error between the two decreases. It is also noticed that away from the peak of the distribution, the statistical curves get more uneven because of the presence of fewer fibers in each interval. Figure 8c is for a higher total shear and it is observed that the error between statistical and analytical solutions for 360 fibers is smaller as compared to that in Fig. 8b, while for 1800 fibers the error is not noticeable.

Figures 9a and 9b show plots of the variation of relative error in standard deviation with number of fibers. The figures show the effect of number of fibers, total shear, and angular intervals on the error $e$. The angular intervals considered for generating the statistical results are 1°. Figure 9a shows the decrease in error with increase in number of fibers, which can also be observed from Figs. 8b and 8c.
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Fig. 9a. Effect of total shear on the relative error in standard deviation.

However, the error is observed to remain constant after a certain number of fibers and found to be significant for $\gamma = 10$. Also as total shear is increased, the initial error value decreases except for $\gamma = 10$ where it remains constant around 10%. To investigate this behavior further, Fig. 9b was drawn for $\gamma = 10$, using smaller angular intervals. It becomes clear from this figure that with higher total shear, smaller values of angular intervals give better approximations.

**Planar Elongational Flow**

Figure 10a shows a plot similar to Fig. 8b having three curves, two of which are from statistical analysis and third is generated from Eq 57. The number of fibers considered are 360 and 1800 for statistical results. It is observed that the curve for 1800 fibers approximates the analytical solution better than 360 fibers. As in the case of simple shear flow, the curve gets very uneven away from its peak showing that fewer fibers are present in that orientation.

Figure 10b shows the variation of relative error in standard deviation against number of fibers. It is interesting to note the similarity between the behavior of error for simple shear and planar elongational flows. In both cases the error decreases with increase in number of fibers and approaches to an asymptotic value which is dependent on the size of the angular interval. The initial value of error is higher for greater total elongation, and higher values of total elongation require smaller angular intervals for accurate results, as seen for $\varepsilon = 2.0$.

The plots depicting the errors for both simple shear and planar elongational flows show that $N = 1000$ can be considered as acceptable number of fibers for the calculation of orientation distribution function from statistical analysis. On the other hand, the optimum size of angular intervals for the statistical solution depends on the strength of the flow field. For high total shear or elongation, smaller angular intervals are needed to minimize error in standard deviation.

**CONCLUSIONS**

An analytical-statistical technique for describing the fiber orientation state of numerous fibers in arbitrary two-dimensional homogeneous flows is presented. The close form solutions of the equation of...
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rotation for a finite aspect ratio fiber is obtained utilizing an equivalent strain tensor expressed in terms of velocity gradients.

The orientation angles of large number of fibers can hence be calculated by the discrete representation of each fiber in terms of an orientation vector. The ability to solve the orientation of individual fibers analytically results in the possibility of generating a statistical orientation distribution function. The accuracy of the statistically generated orientation distribution function is found to be dependent on the number of fibers utilized in the analytical solution. The results obtained by using different number of fibers are compared with the exact solution and the error in standard deviation is found to decrease by increasing the number of fibers. In addition, for high total shear or elongation values, the use of smaller angular intervals substantially improves the accuracy of the generated orientation distribution function. Examples are presented for numerous homogeneous flows which serve as a check for the validity of this method. It is apparent that the generality of the derivation enables one to extend the present analysis to three-dimensional orientation and homogeneous flow fields. This technique can also be easily implemented to complex flows by solving the orientation state along the streamlines utilizing the numerically obtained velocity gradients.

REFERENCES
