ECE 2713
Test 1

Tuesday, March 25, 2025
12:00 PM - 1:15 PM

Spring 2025 | Name: SOLUTION

Dr. Havlicek | Student Num: ________

Directions: This test is closed book and closed notes. Calculators are allowed. You may also use the formula sheet provided with the test. All work must be your own. You have 75 minutes to complete the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

TOTAL (100):

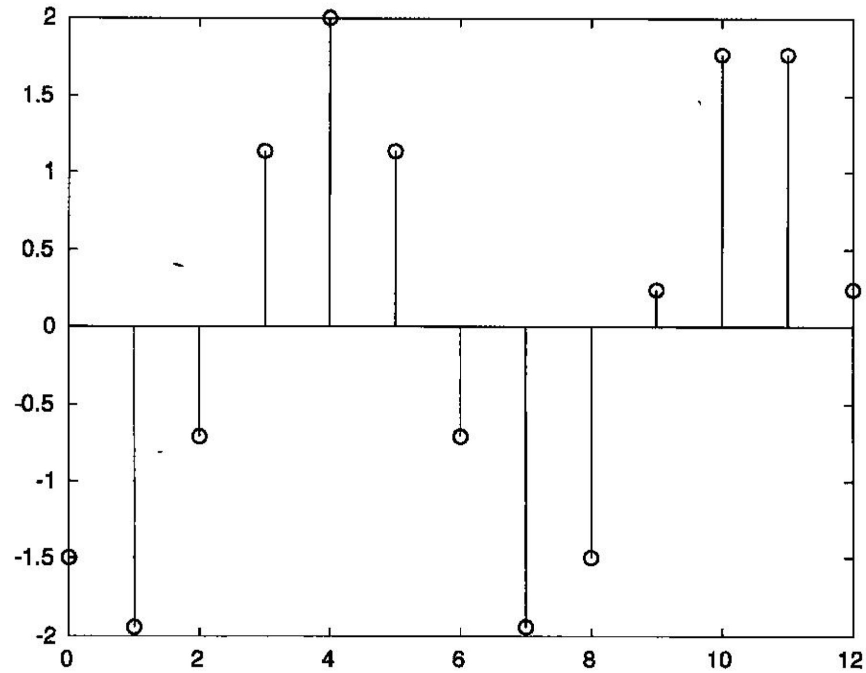
____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ________ Date: ________

1. **25 pts**. A **periodic** discrete-time sinusoidal signal $x[n]$ is given by $x[n] = A\cos(\omega_0 n + \phi)$.

The figure below shows a graph of ***exactly one period:***



Notice that the graph starts at $n = 0$ and ends at $n = 12$. This is exactly one period.

Find the amplitude A, frequency ω_0, and initial phase ϕ.

Hint: according to the formula sheet, if a discrete-time sinusoidal signal $x[n]$ is periodic, then $\frac{\omega_0}{2\pi} = \frac{m}{N}$ where m and N are integers.

Max amplitude is $|x[n]| = 2$ when $n=4$ $\Rightarrow$ $\boxed{A=2}$

The graph shows 13 samples and this is given to be exactly one period... $\Rightarrow$ $N = 13$.

The graph "goes around" twice in one period, so $m=2$.

$$\frac{\omega_0}{2\pi} = \frac{m}{N} = \frac{2}{13}$$

$$\boxed{\omega_0 = \frac{4\pi}{13}}$$

$$\boxed{\omega_0 = 0.9666}$$

Because the max amplitude occurs when $n=4$, the argument of $\cos(\cdot)$ must be zero (modulo 2π) when $n=4$:

$\omega_0 n + \phi = 0$ when $n=4$

$\frac{16\pi}{13} + \phi = 0$

$$\boxed{\phi = -16\pi/13}$$

$$\boxed{\phi = -3.86658}$$

2. **25 pts.** A continuous-time signal $x(t)$ is given by

$$x(t) = 5\cos\left(\frac{\pi}{17}t + \frac{\pi}{4}\right) + 7\cos\left(\frac{\pi}{17}t + \frac{\pi}{8}\right).$$

Use phasor addition to express $x(t)$ in the form

$$x(t) = A\cos\left(\frac{\pi}{17}t + \phi\right).$$

Phasor for $5\cos\left[\frac{\pi}{17}t + \frac{\pi}{4}\right]$: $X_1 = 5e^{j\pi/4}$

Phasor for $7\cos\left[\frac{\pi}{17}t + \frac{\pi}{8}\right]$: $X_2 = 7e^{j\pi/8}$

Phasor for $x(t)$:

$$\begin{aligned} X &= X_1 + X_2 \\ &= 5\cos\tfrac{\pi}{4} + j5\sin\tfrac{\pi}{4} + 7\cos\tfrac{\pi}{8} + j7\sin\tfrac{\pi}{8} \\ &= 3.53553 + j3.53553 + 6.46716 + j2.67878 \\ &= 10.0027 + j6.21432 \end{aligned}$$

$$\begin{aligned} |X| &= \sqrt{10.0027^2 + 6.21432^2} \\ &= \sqrt{100.054 + 38.6177} \\ &= \sqrt{138.672} \\ &= 11.7759 \end{aligned}$$

$\Rightarrow$ Since X lies in the first quadrant, no angle correction is needed.

$$\begin{aligned} \phi &= \arctan \frac{6.21432}{10.0027} \\ &= \arctan[0.621265] \\ &= 0.555909 \end{aligned}$$

$$X = 11.7759e^{j0.555909}$$

$$\boxed{x(t) = 11.7759\cos\left[\frac{\pi}{17}t + 0.555909\right]}$$

More Workspace for Problem 2...

3. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2].$$

The system input is given by

$$x[n] = 2\delta[n+1] - 4\delta[n] + 2\delta[n-1].$$

Find the system output $y[n]$.

$$y[n] = x[n] * h[n] = x[n] * \left(\delta[n] + 2\delta[n-1] + 3\delta[n-2]\right)$$

$$= x[n] + 2x[n-1] + 3x[n-2]$$

$$= 2\delta[n+1] - 4\delta[n] + 2\delta[n-1]$$

$$+ 4\delta[n] - 8\delta[n-1] + 4\delta[n-2]$$

$$+ 6\delta[n-1] - 12\delta[n-2] + 6\delta[n-3]$$

$$y[n] = 2\delta[n+1] + 0\,\delta[n] + 0\,\delta[n-1] - 8\delta[n-2] + 6\delta[n-3]$$

$$\boxed{y[n] = 2\delta[n+1] - 8\delta[n-2] + 6\delta[n-3]}$$

3: "OTHER WAY":

$$y[n] = h[n] * x[n]$$
$$= h[n] * \left(2\delta[n+1] - 4\delta[n] + 2\delta[n-1]\right)$$
$$= 2h[n+1] - 4h[n] + 2h[n-1]$$
$$= 2\delta[n+1] + 4\delta[n] + 6\delta[n-1]$$
$$- 4\delta[n] - 8\delta[n-1] - 12\delta[n-2]$$
$$+ 2\delta[n-1] + 4\delta[n-2] + 6\delta[n-3]$$

$$y[n] = 2\delta[n+1] + 0\,\delta[n] + 0\,\delta[n-1] - 8\delta[n-2] + 6\delta[n-3]$$

$$\boxed{y[n] = 2\delta[n+1] - 8\delta[n-2] + 6\delta[n-3]}$$

4. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{6}\right)^n u[n].$$

The system input is given by

$$x[n] = \left(-\frac{1}{3}\right)^n \left(u[n+2] - u[n-3]\right) = \begin{cases} \left(-\frac{1}{3}\right)^n, & -2 \le n \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

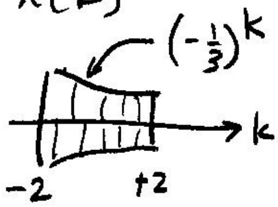
Find the system output $y[n]$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

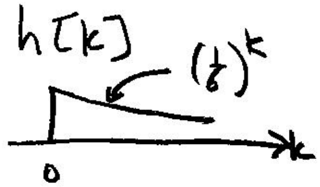
Hint: here are the steps for performing convolution:

1. write $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.
2. Use the definition of $x[n]$ given above to draw the graph of $x[k]$.
3. Use the definition of $h[n]$ given above to draw the graph of $h[k]$.
4. Slide the graph of $h[k]$ to the right by $-n$ to get the graph of $h[k-(-n)] = h[n+k]$.
5. Flip the graph of $h[n+k]$ with respect to k to get the graph of $h[-k-(-n)] = h[n-k]$.
6. For the n's in each region, multiply the graph of $x[k]$ with the graph of $h[n-k]$ and add up the product graph to get $y[n]$.

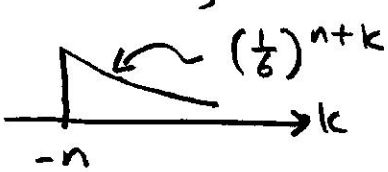
$x[k]$



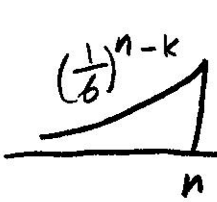
$h[k]$



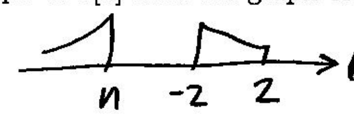
$h[k--n] = h[n+k]$



$h[-k--n] = h[n-k]$

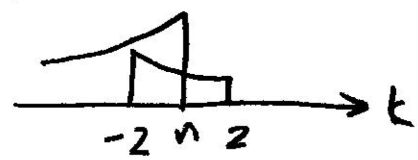


case I) $n < -2$:



: $y[n] = 0$

case II) $-2 \le n < 2$:



$$y[n] = \sum_{k=-2}^{n} \left(-\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^{n-k} = \left(\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} \left(-\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^{-k}$$

$$= \left(\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} \left(-\tfrac{1}{3}\right)^k 6^k = \left(\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} \left(-\tfrac{6}{3}\right)^k = \left(\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} (-2)^k$$

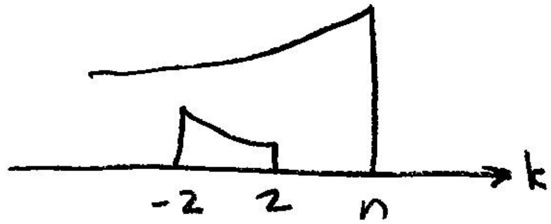
$$= \left(\tfrac{1}{6}\right)^n \frac{(-2)^{-2} - (-2)^{n+1}}{1+2} = \frac{\left(\tfrac{1}{6}\right)^n \left[\tfrac{1}{4} + 2(-2)^n\right]}{3}$$

$$= \tfrac{1}{3}\left(\tfrac{1}{6}\right)^n \left(\tfrac{1}{4}\right) + \tfrac{1}{3}\left(\tfrac{1}{6}\right)^n 2(-2)^n$$

$$= \tfrac{1}{12}\left(\tfrac{1}{6}\right)^n + \tfrac{2}{3}\left(\tfrac{-2}{6}\right)^n = \tfrac{1}{12}\left(\tfrac{1}{6}\right)^n + \tfrac{2}{3}\left(-\tfrac{1}{3}\right)^n$$

More Workspace for Problem 4...

Case III) $n \geq 2$



$$y[n] = \sum_{k=-2}^{2} \left(-\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^{n-k}$$

$$= \left(\tfrac{1}{6}\right)^n \sum_{k=-2}^{2} \left(-\tfrac{1}{3}\right)^k 6^k = \left(\tfrac{1}{6}\right)^n \sum_{k=-2}^{2} (-2)^k$$

$$= \left(\tfrac{1}{6}\right)^n \frac{(-2)^{-2} - (-2)^3}{1+2} = \left(\tfrac{1}{6}\right)^n \frac{\tfrac{1}{4} + 8}{3}$$

$$= \tfrac{1}{3}\left(\tfrac{1}{6}\right)^n \left[\tfrac{1}{4} + \tfrac{32}{4}\right] = \tfrac{1}{3}\left(\tfrac{1}{6}\right)^n \tfrac{33}{4} = \tfrac{11}{4}\left(\tfrac{1}{6}\right)^n$$

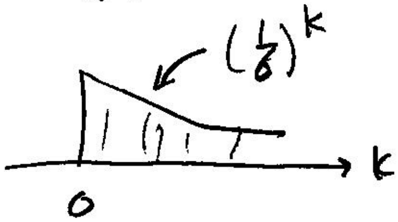
All Together:

$$y[n] = \begin{cases} 0 & , \; n < -2 \\ \tfrac{1}{12}\left(\tfrac{1}{6}\right)^n + \tfrac{2}{3}\left(-\tfrac{1}{3}\right)^n & , \; -2 \leq n < 2 \\ \tfrac{11}{4}\left(\tfrac{1}{6}\right)^n & , \; n \geq -2 \end{cases}$$

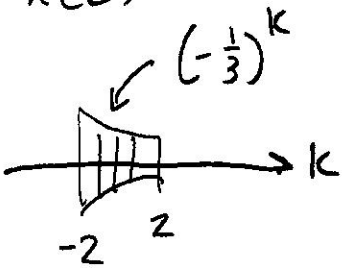
4: "OTHER WAY": $y[n] = x[n] * h[n]$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

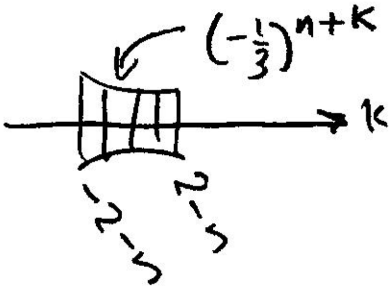
$h[k]$



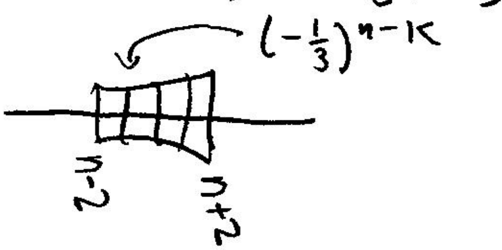
$x[k]$



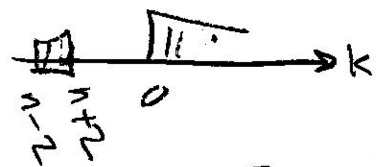
$x[k - -n] = x[n+k]$



$x[-k - -n] = x[n-k]$



case I)

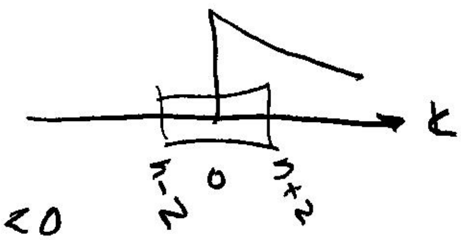


$n+2 < 0$

$n < -2$

$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

case II)



$n+2 \geq 0$

and $n-2 < 0$

$n \geq -2$ and $n < 2 \longleftrightarrow -2 \leq n < 2$

$$y[n] = \sum_{k=0}^{n+2} \left(\frac{1}{6}\right)^k \left(-\frac{1}{3}\right)^{n-k}$$

$$= \sum_{k=0}^{n+2} \left(\frac{1}{6}\right)^k \left(-\frac{1}{3}\right)^n \left(-\frac{1}{3}\right)^{-k} = \left(-\frac{1}{3}\right)^n \sum_{k=0}^{n+2} \left(\frac{1}{6}\right)^k (-3)^k$$

$$= \left(-\frac{1}{3}\right)^n \sum_{k=0}^{n+2} \left(-\frac{1}{2}\right)^k = \left(-\frac{1}{3}\right)^n \frac{\left(-\frac{1}{2}\right)^0 - \left(-\frac{1}{2}\right)^{n+3}}{1 + 1/2}$$

$$= \left(-\frac{1}{3}\right)^n \frac{1 - \left(-\frac{1}{2}\right)^n \left(-\frac{1}{2}\right)^3}{3/2}$$

$$= \left(-\frac{1}{3}\right)^n \left(\frac{2}{3}\right) \left[1 + \frac{1}{8}\left(-\frac{1}{2}\right)^n\right]$$

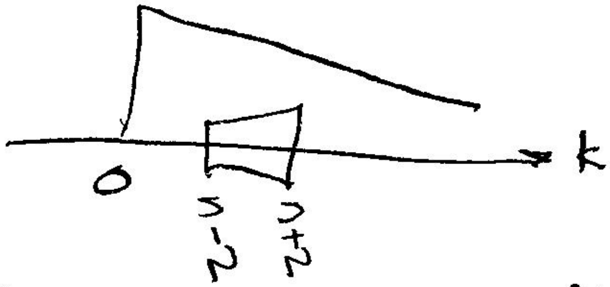
$$= \frac{2}{3}\left(-\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)\left(\frac{1}{8}\right)\left(-\frac{1}{3}\right)^n \left(-\frac{1}{2}\right)^n$$

$$= \frac{2}{3}\left(-\frac{1}{3}\right)^n + \frac{2}{24}\left(\frac{1}{6}\right)^n$$

$$= \frac{2}{3}\left(-\frac{1}{3}\right)^n + \frac{1}{12}\left(\frac{1}{6}\right)^n$$

4: "OTHER WAY" . . .

Case III)



$n-2 \geq 0$

$n \geq 2$

$$y[n] = \sum_{k=n-2}^{n+2} \left(\tfrac{1}{6}\right)^k \left(-\tfrac{1}{3}\right)^{n-k} = \left(-\tfrac{1}{3}\right)^n \sum_{k=n-2}^{n+2} \left(\tfrac{1}{6}\right)^k (-3)^k$$

$$= \left(-\tfrac{1}{3}\right)^n \sum_{k=n-2}^{n+2} \left(-\tfrac{1}{2}\right)^k = \left(-\tfrac{1}{3}\right)^n \frac{\left(-\tfrac{1}{2}\right)^{n-2} - \left(-\tfrac{1}{2}\right)^{n+3}}{\underbrace{1 + 1/2}_{3/2}}$$

$$= \left(-\tfrac{1}{3}\right)^n \left(\tfrac{2}{3}\right)\left[\left(-\tfrac{1}{2}\right)^n \left(-\tfrac{1}{2}\right)^{-2} - \left(-\tfrac{1}{2}\right)^n \left(-\tfrac{1}{2}\right)^3\right]$$

$$= \left(-\tfrac{1}{3}\right)^n \left(-\tfrac{1}{2}\right)^n \left(\tfrac{2}{3}\right)\left[(-2)^2 + \tfrac{1}{8}\right] = \tfrac{2}{3}\left(\tfrac{1}{6}\right)^n \left[4 + \tfrac{1}{8}\right]$$

$$= \tfrac{2}{3}\left(\tfrac{1}{6}\right)^n \tfrac{33}{8} = \tfrac{11}{4}\left(\tfrac{1}{6}\right)^n$$

All Together

$$y[n] = \begin{cases} 0 & , \ n < -2 \\ \tfrac{2}{3}\left(-\tfrac{1}{3}\right)^n + \tfrac{1}{12}\left(\tfrac{1}{6}\right)^n & , \ -2 \leq n < 2 \\ \tfrac{11}{4}\left(\tfrac{1}{6}\right)^n & , \ n \geq 2 \end{cases}$$