ECE 2713
Test 2

Tuesday, April 28, 2026
12:00 PM - 1:15 PM

Spring 2026 — Name: SOLUTION

Dr. Havlicek — Student Num:____________________

Directions: This test is closed book and closed notes. Calculators are allowed. You may also use the formula sheet provided with the test. Other materials are not allowed. All work must be your own. You have 75 minutes to complete the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

____

TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:____________________________ Date:____________________________

1. **25 pts**. A discrete-time LTI system H has impulse response

$$h[n] = \left(-\frac{1}{4}\right)^n u[n] \xleftrightarrow{\text{Table}} H(e^{j\omega}) = \frac{1}{1+\frac{1}{4}e^{-j\omega}}$$

and input

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1].$$

$$\text{Table + Time Shift} \longrightarrow X(e^{j\omega}) = \frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

Use the discrete-time Fourier transform (DTFT) to find the output signal $y[n]$.

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{4}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1+\frac{1}{4}e^{-j\omega}}$$

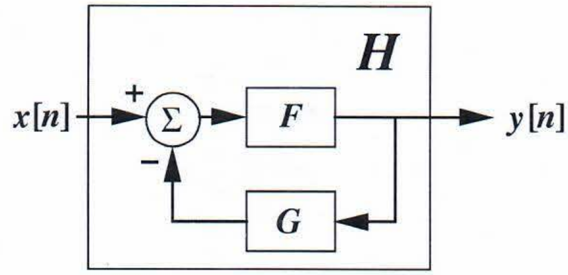
$$A = \left.\frac{\theta}{1+\frac{1}{4}\theta}\right|_{\theta=2} = \frac{2}{1+1/2} = \frac{2}{3/2} = \frac{4}{3}$$

$$B = \left.\frac{\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-4} = \frac{-4}{1+2} = -\frac{4}{3}$$

$$Y(e^{j\omega}) = \frac{4/3}{1-\frac{1}{2}e^{-j\omega}} - \frac{4/3}{1+\frac{1}{4}e^{-j\omega}}$$

Table: $\boxed{y[n] = \frac{4}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}\left(-\frac{1}{4}\right)^n u[n]}$

2. **25 pts**. Consider the discrete-time LTI system H shown below.



F and G are both discrete-time LTI systems.

The impulse response of F is given by $f[n] = \left(\frac{1}{4}\right)^n u[n]$. $\overset{\text{Table}}{\longleftrightarrow}$ $F(e^{j\omega}) = \dfrac{1}{1-\frac{1}{4}e^{-j\omega}}$

The impulse response of G is given by $g[n] = \left(\frac{1}{3}\right)^{n-1} u[n-1]$.

Table, Time Shift $\rightarrow$ $G(e^{j\omega}) = \dfrac{e^{-j\omega}}{1-\frac{1}{3}e^{-j\omega}}$

(a) **11 pts**. Find the overall system frequency response $H(e^{j\omega})$.

$$H(e^{j\omega}) = \frac{F(e^{j\omega})}{1+F(e^{j\omega})G(e^{j\omega})} = \frac{\dfrac{1}{1-\frac{1}{4}e^{-j\omega}}}{1+\dfrac{e^{-j\omega}}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}}$$

$$H(e^{j\omega}) \times \underbrace{\frac{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}}_{\text{ONE}} = \frac{1-\frac{1}{3}e^{-j\omega}}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})+e^{-j\omega}}$$

$$= \frac{1-\frac{1}{3}e^{-j\omega}}{1-(\frac{1}{4}+\frac{1}{3})e^{-j\omega}+\frac{1}{12}e^{-j2\omega}+e^{-j\omega}}$$

$$= \frac{1-\frac{1}{3}e^{-j\omega}}{1-\frac{7}{12}e^{-j\omega}+e^{-j\omega}+\frac{1}{12}e^{-j2\omega}} \longmapsto$$

Problem 2, cont...

$$\boxed{H(e^{j\omega}) = \frac{1 - \frac{1}{3}e^{-j\omega}}{1 + \frac{5}{12}e^{-j\omega} + \frac{1}{12}e^{-j2\omega}}}$$

(b) **11 pts**. Find the difference equation (I/O equation) that relates the input signal $x[n]$ and output signal $y[n]$ of the overall system H.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{3}e^{-j\omega}}{1 + \frac{5}{12}e^{-j\omega} + \frac{1}{12}e^{-j2\omega}}$$

$$Y(e^{j\omega})\left[1 + \tfrac{5}{12}e^{-j\omega} + \tfrac{1}{12}e^{-j2\omega}\right] = X(e^{j\omega})\left[1 - \tfrac{1}{3}e^{-j\omega}\right]$$

$$Y(e^{j\omega}) + \tfrac{5}{12}e^{-j\omega}Y(e^{j\omega}) + \tfrac{1}{12}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega}) - \tfrac{1}{3}e^{-j\omega}X(e^{j\omega})$$

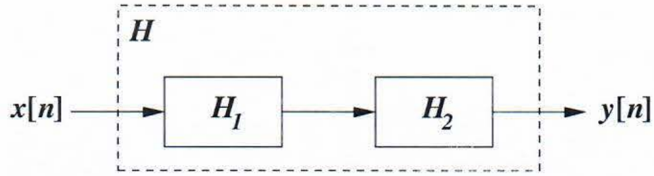
DTFT^{-1}:

$$\boxed{y[n] + \tfrac{5}{12}y[n-1] + \tfrac{1}{12}y[n-2] = x[n] - \tfrac{1}{3}x[n-1]}$$

(c) **3 pts**. Is H an FIR digital filter or an IIR digital filter? Briefly justify your answer.

H IS AN IIR digital filter because the I/O equation has shifts of $y[n]$ (like $y[n-1]$ and $y[n-2]$).

3. **25 pts.** The discrete-time LTI system H is formed by a series connection of two discrete-time LTI systems H_1 and H_2 as shown in the figure below:



The impulse response of LTI system H_1 is given by Table

$$h_1[n] = 2\left(\frac{1}{2}\right)^n u[n]. \longleftrightarrow H_1(e^{j\omega}) = \frac{2}{1-\frac{1}{2}e^{-j\omega}}$$

The difference equation (I/O equation) that relates input signal $x[n]$ and output signal $y[n]$ of the overall system H is given by

$$y[n] - \frac{7}{10}y[n-1] + \frac{1}{10}y[n-2] = x[n].$$

(a) **8 pts.** Find the system frequency response $H(e^{j\omega})$.

DTFT: $Y(e^{j\omega}) - \frac{7}{10}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{10}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega})$

$$Y(e^{j\omega})\left[1 - \frac{7}{10}e^{-j\omega} + \frac{1}{10}e^{-j2\omega}\right] = X(e^{j\omega})$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{7}{10}e^{-j\omega}+\frac{1}{10}e^{-j2\omega}}}$$

Problem 3, cont...

(b) **10 pts.** Find the impulse response $h_2[n]$ of LTI system H_2.

$$(*) \quad H(e^{j\omega}) = \frac{1}{1-\frac{7}{10}e^{-j\omega}+\frac{1}{10}e^{-j2\omega}} = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{5}e^{-j\omega}\right)}$$

Also, because it is a <u>series</u> connection,

$$(**) \quad H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}) = \frac{2}{1-\frac{1}{2}e^{j\omega}} H_2(e^{j\omega})$$

Equating $(*)$ and $(**)$, we have

$$H(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{5}e^{-j\omega}\right)} = \frac{2H_2(e^{j\omega})}{1-\frac{1}{2}e^{-j\omega}}$$

$$2H_2(e^{j\omega}) = \frac{1}{1-\frac{1}{5}e^{-j\omega}}$$

$$H_2(e^{j\omega}) = \frac{1/2}{1-\frac{1}{5}e^{-j\omega}}$$

Table: $\boxed{h_2[n] = \frac{1}{2}\left(\frac{1}{5}\right)^n u[n]}$

Problem 3, cont...

(c) **7 pts**. Is H_2 a *stable* discrete-time LTI system? Justify your answer.

A discrete-time LTI system is stable iff the impulse response is absolutely summable.

i.e., if and only if $\sum_{n=-\infty}^{\infty} |h_2[n]| < \infty$.

$$\Rightarrow \sum_{n=-\infty}^{\infty} |h_2[n]| = \sum_{n=-\infty}^{\infty} \left|\tfrac{1}{2}\left(\tfrac{1}{5}\right)^n u[n]\right| = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n$$

$$\underset{\text{sheet}}{\overset{\text{formula}}{=}} \frac{1}{2}\ \frac{1}{1-\frac{1}{5}} = \frac{1}{2}\ \frac{1}{4/5}$$

$$= \frac{1}{2}\cdot\frac{5}{4} = \frac{5}{8} < \infty$$

$$\boxed{\Rightarrow H_2 \text{ is stable because } \sum_{n=-\infty}^{\infty} |h_2[n]| < \infty}$$

4. **25 pts**. Consider a digital filter H with difference equation (I/O equation) given by

$$y[n] = x[n] - 2x[n-1] + 2x[n-3] - x[n-4].$$

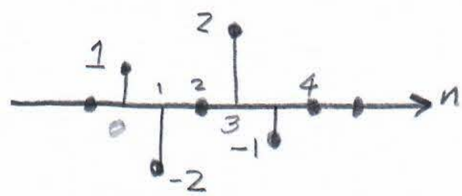
(a) **3 pts**. Is H an FIR digital filter or an IIR digital filter? Briefly justify your answer.

H is an FIR filter because the I/O equation has no shifts of $y[n]$.

(b) **6 pts**. Find the system impulse response $h[n]$.

It follows immediately from "convolution with deltas" that

$$\boxed{h[n] = \delta[n] - 2\delta[n-1] + 2\delta[n-3] - \delta[n-4]}$$



odd symmetry about midpoint $\Rightarrow$ LINEAR PHASE FIR

(c) **6 pts**. Find the filter frequency response $H(e^{j\omega})$.

Table + Time Shift Property:

$$\boxed{H(e^{j\omega}) = 1 - 2e^{-j\omega} + 2e^{-j3\omega} - e^{-j4\omega}}$$

Problem 4, cont...

(d) **10 pts**. Find the filter magnitude response $|H(e^{j\omega})|$ and phase response $\arg H(e^{j\omega})$.

Hint 1: use the "linear phase trick" to write $H(e^{j\omega})$ in polar form and then simply read off the magnitude response and phase response functions.

Hint 2: remember to use Euler's formula for **sine** if the symmetry is **odd**.

Highest Power of the character = 4 $\Rightarrow$ Factor out $e^{-j2\omega}$

$$H(e^{j\omega}) = \left[e^{j2\omega} - 2e^{j\omega} + 2e^{-j\omega} - e^{-j2\omega}\right]e^{-j2\omega}$$

$$= \left[(e^{j2\omega} - e^{-j2\omega}) - 2(e^{j\omega} - e^{-j\omega})\right]e^{-j2\omega}$$

$$= \left[2j\,\frac{e^{j2\omega} - e^{-j2\omega}}{2j} - 2\cdot 2j\,\frac{e^{j\omega} - e^{-j2\omega}}{2j}\right]e^{-j2\omega}$$

$$= \left[2j\sin 2\omega - 4j\sin\omega\right]e^{-j2\omega}$$

$$\Rightarrow j = e^{j\pi/2}$$

$$= \left[2\sin 2\omega - 4\sin\omega\right]e^{j\pi/2}e^{-j2\omega}$$

$$= \left[2\sin 2\omega - 4\sin\omega\right]e^{j(\pi/2 - 2\omega)}$$

$$\boxed{\begin{aligned} |H(e^{j\omega})| &= 2\sin 2\omega - 4\sin\omega \\ \arg H(e^{j\omega}) &= \pi/2 - 2\omega \end{aligned}}$$

Note that this is generalized linear phase and the spectral "magnitude" $|H(e^{j\omega})|$ is negative for some ω... see notes pp. 5.85–5.87