ECE 2713

HW 7 SOLUTION

HAVLICEK

① P-9.3) H is an LTI system with transfer function

$$H(z) = 1 + 5z^{-2} - 3z^{-3} + 2z^{-5} + 4z^{-7}$$

Note: we know that this is an FIR filter because the denominator of $H(z)$ is 1.

a) Find the difference equation (I/O equation):

$$H(z) = \frac{Y(z)}{X(z)} = 1 + 5z^{-2} - 3z^{-3} + 2z^{-5} + 4z^{-7}$$

$$Y(z) = X(z)\left[1 + 5z^{-2} - 3z^{-3} + 2z^{-5} + 4z^{-7}\right]$$

$$Y(z) = X(z) + 5z^{-2}X(z) - 3z^{-3}X(z) + 2z^{-5}X(z) + 4z^{-7}X(z)$$

→ We don't need to know the ROC of $H(z)$ to invert this.
→ All we need is the time shift property:

$$z^{-n_0}X(z) \overset{z}{\longleftrightarrow} x[n-n_0]$$

⇒ And we don't know what $X(z)$ and $Y(z)$ are... they could be anything. But whatever they are, we do know that

$$X(z) \overset{z}{\longleftrightarrow} x[n] \quad \text{and} \quad Y(z) \overset{z}{\longleftrightarrow} y[n]$$

So, taking the inverse z-transform on both sides, we get:

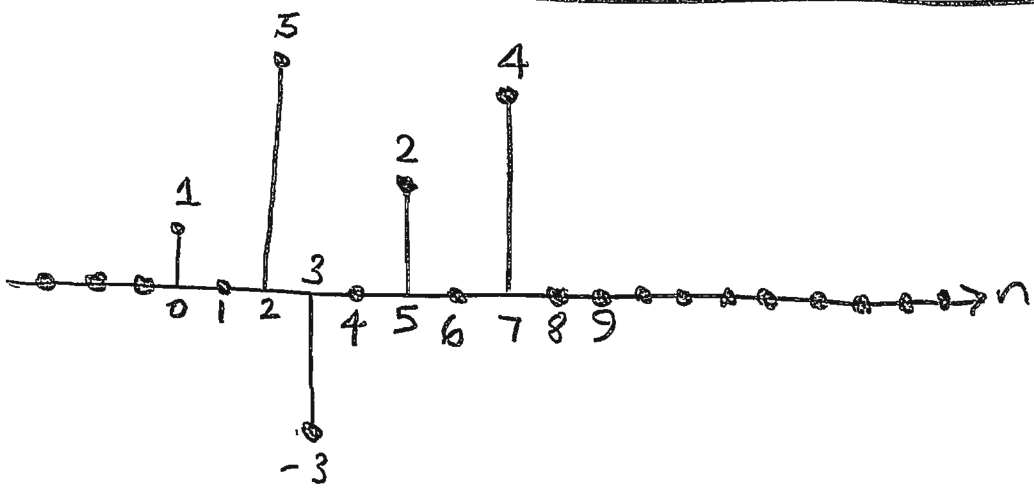
$$\boxed{y[n] = x[n] + 5x[n-2] - 3x[n-3] + 2x[n-5] + 4x[n-7]}$$

b) When the input signal is $x[n] = \delta[n]$, the output signal is the impulse response $h[n]$.

→ So what this part is really asking for is just the impulse response $h[n]$.

→ Plugging in $x[n] = \delta[n]$ to the answer from part (a), we get

$$\boxed{h[n] = \delta[n] + 5\delta[n-2] - 3\delta[n-3] + 2\delta[n-5] + 4\delta[n-7]}$$



Note: since we know that H is an FIR filter (because the denominator of $H(z)$ is one), we know that $h[n]$ has to be a finite-length signal.

⇒ This means that the ROC of $H(z)$ must be all z except possibly $z=0$ and $z=\infty$, which must always be checked separately. Checking them, we find that $H(0) \to \infty$ and $H(\infty) = 1$. So $z=0$ is a pole. This means the ROC of $H(z)$ is $|z| > 0$.

⇒ So we could alternatively use the Table to find $h[n]$ from $H(z)$ directly... you get the same answer.

② P-9.8) H is an LTI system with I/O equation: 2-1

$$y[n] = x[n] - x[n-1] + x[n-2] - x[n-3] + x[n-4].$$

a) Find $h[n]$. By definition, $h[n]$ is the output when the input is $x[n] = \delta[n]$. Plugging this into the given I/O equation, we get

$$\boxed{h[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3] + \delta[n-4]}$$

b) Find $H(z)$. Using the Table plus the time shift property, we get

$$\boxed{H(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4}, \quad |z| > 0}$$

⇒ The Table says $\delta[n] \overset{z}{\longleftrightarrow} 1$, <u>all z</u>. But remember that you always have to <u>check</u> the two points $z=0$ and $z=\infty$. For this $H(z)$, we get $H(0) \to \infty$ and $H(\infty) \to 1$. So there is a pole at $z=0$ ... it's actually a 4th-order pole. This means the ROC of $H(z)$ is all z <u>except</u> $z=0$. In other words, the ROC is $|z|>0$ as stated in my answer above.

d) Find the frequency response $H(e^{j\omega})$.

$\Rightarrow$ $H(e^{j\omega})$ is given by $H(z)$ on the unit circle of the z-plane. In other words, $H(e^{j\omega})$ is obtained from $H(z)$ by just looking at the z's that have unit magnitude. On the unit circle, $z = e^{j\omega}$. So all we have to do for this part is plug in $z = e^{j\omega}$ to $H(z)$...

$\Rightarrow$ Provided that $H(e^{j\omega})$ exists!!!

$\Rightarrow$ In other words, provided that the ROC of $H(z)$ includes the unit circle!!!

CHECK: ROC is $|z| > 0$

$\rightarrow$ includes the unit circle ✓

$$\boxed{H(e^{j\omega}) = 1 - e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega} + e^{-j4\omega}}$$

Note: In this problem, we see that H is an FIR filter (because the denominator of $H(z)$ is one).

$\rightarrow$ Also, from part (a), we see that $h[n]$ has even symmetry about the middle sample.

$\Rightarrow$ So H is a linear phase FIR filter.

$\longrightarrow$

... This means that we can apply the "half-power linear phase trick" to simplify $H(e^{j\omega})$:

highest power: $e^{-j4\omega}$

half the highest power: $e^{-j2\omega}$

→ factor out $e^{-j2\omega}$

$$H(e^{j\omega}) = \left[e^{j2\omega} - e^{j\omega} + 1 - e^{-j\omega} + e^{-j2\omega}\right] e^{-j2\omega}$$

$$= \left[1 - (e^{j\omega} + e^{-j\omega}) + (e^{j2\omega} + e^{-j2\omega})\right] e^{-j2\omega}$$

$$H(e^{j\omega}) = \left[1 - 2\cos\omega + 2\cos 2\omega\right] e^{-j2\omega}$$

③ H is an LTI system with I/O equation $\boxed{3-1}$

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

a) Find $H(z)$: Take z-transform on both sides:

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1}\right] = X(z)\left[1 + \frac{1}{3}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} \quad //$$

b) Give a pole zero plot for $H(z)$:

→ The numerator has a root at $z = -\frac{1}{3}$. This is a <u>zero</u> of $H(z)$.

→ The denominator has a root at $z = \frac{1}{2}$. This is a <u>pole</u> of $H(z)$.

→ we must also check the points $z=0$ and $z \to \infty$.

$$\lim_{z\to 0} H(z) = \lim_{z\to 0} \frac{1+\frac{1}{3}\frac{1}{z}}{1-\frac{1}{2}\frac{1}{z}} = \lim_{\theta\to\infty} \frac{1+\frac{1}{3}\theta}{1-\frac{1}{2}\theta} \longrightarrow \frac{\infty}{\infty}$$

It's indeterminate, so use L'Hopital's rule--

$$\lim_{\theta\to\infty} \frac{1+\frac{1}{3}\theta}{1-\frac{1}{2}\theta} = \lim_{\theta\to\infty} \frac{\frac{\partial}{\partial\theta}\left(1+\frac{1}{3}\theta\right)}{\frac{\partial}{\partial\theta}\left(1-\frac{1}{2}\theta\right)} = \lim_{\theta\to\infty} \frac{1/3}{-1/2} = -\frac{2}{3}$$

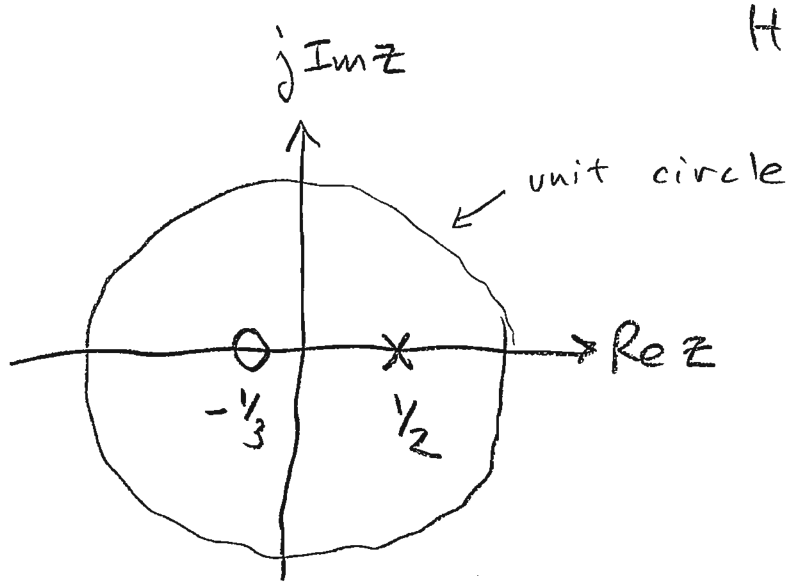
So $z=0$ is not a pole nor a zero of $H(z)$.

$\boxed{3-2}$

$$\lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{1+\frac{1}{3}\frac{1}{z}}{1-\frac{1}{2}\frac{1}{z}} = \frac{1}{1} = 1$$

$z=\infty$ is also neither a pole nor a zero of $H(z)$.

P-Z plot:



c) Assume $H(e^{j\omega})$ exists. Well, $H(e^{j\omega})$ is given by $H(z)$ for $z=e^{j\omega}$ ... in other words, $H(z)$ <u>is</u> $H(e^{j\omega})$ above the unit circle of the z-plane.

So the fact that $H(e^{j\omega})$ exists means:

~~...~~ The unit circle <u>must</u> be in the ROC of $H(z)$.

ROC: $|z| > \frac{1}{2}$

c)... Now, partial fractions does not work unless the fraction is strictly proper.

→ That means the order of the numerator has to be strictly less than the order of the denominator.

→ But here, we've got

$$H(z) = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

⇒ Numerator and denominator are both first order in z^{-1}.

⇒ THIS IS AN IMPROPER FRACTION!

→ You can't apply PFE directly... you will get the wrong answer...

⟶

c)... but luckily there is only one term downstairs... so there is an easy trick that will work:

$$H(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{1}{3}z^{-1}\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}$$

The first term is in your table:

$$\frac{1}{1-\frac{1}{2}z^{-1}},\quad |z|>\frac{1}{2} \quad \overset{Z}{\longleftrightarrow} \quad \left(\frac{1}{2}\right)^n u[n]$$

For the 2nd term, apply the time shift property:

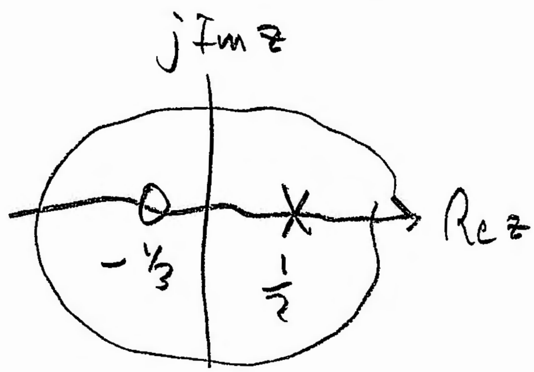
$$\frac{1}{3}z^{-1}\frac{1}{1-\frac{1}{2}z^{-1}},\quad |z|>\frac{1}{2} \quad \overset{Z}{\longleftrightarrow} \quad \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$\boxed{h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]}$$

d) under the assumption of part (c),

- Is the system <u>causal</u>? <u>YES</u>, because $h[n]=0 \;\forall n<0$.
- Is the system <u>stable</u>? <u>YES</u>, because the ROC of $H(z)$ includes the unit circle.

e) Now assume that H is <u>unstable</u> and <u>not causal</u>. Find the ROC of $H(z)$ and find $h[n]$.



- Since H is <u>unstable</u>, the ROC <u>can not</u> contain the unit circle.
- Since H is <u>not causal</u>, the ROC can <u>not</u> be exterior.
- Since there's only one pole, the ROC also can't be annular.

$\rightarrow$ It must be interior. ROC: $|z| < \frac{1}{2}$

So, using the same trick as in part (c), but with our new ROC, we've got

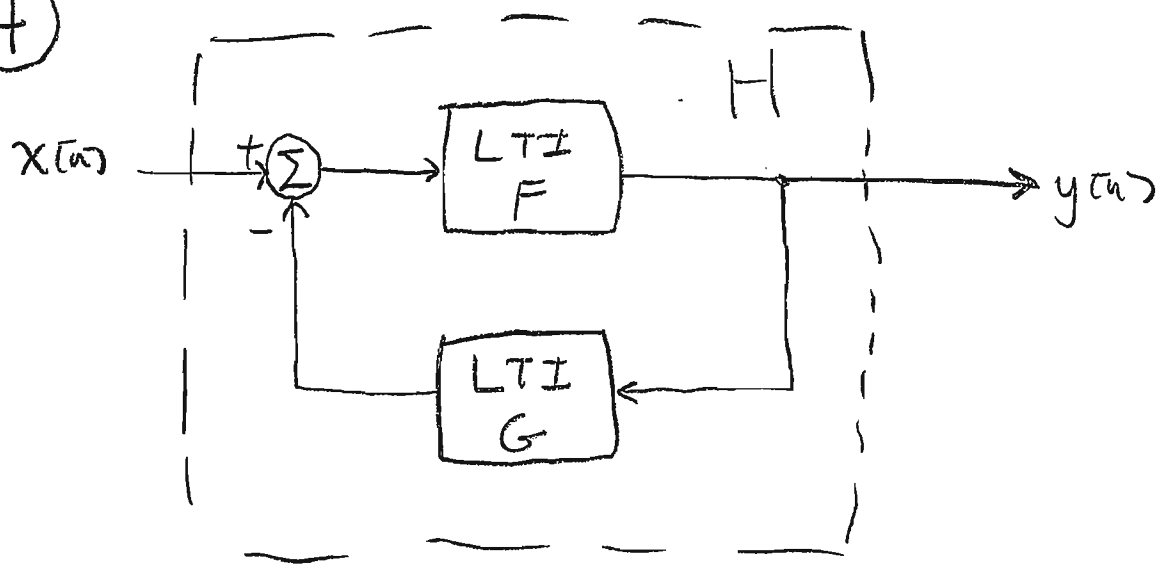
$$H(z) = \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|<\frac{1}{2}} + \underbrace{\frac{1}{3}z^{-1}\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|<\frac{1}{2}}$$

Table:

$$h[n] = -\left(\tfrac{1}{2}\right)^{n} u[-n-1] - \tfrac{1}{3}\left(\tfrac{1}{2}\right)^{n-1} u[-(n-1)-1]$$

$$\boxed{h[n] = -\left(\tfrac{1}{2}\right)^{n} u[-n-1] - \tfrac{1}{3}\left(\tfrac{1}{2}\right)^{n-1} u[-n]}$$

④



H is given to be LTI and causal.

$g[n] = \frac{3}{2}\delta[n-1]$.

→ Since $g[n] = 0 \quad \forall n < 0$, G is causal.

→ This together with the fact that H is causal (given) means that F must also be causal.

- When $x[n] = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1]$, we are given that $y[n] = n\left(\frac{1}{2}\right)^{n} u[n]$

→

a) Find $H(z)$:

$\boxed{4\text{-}2}$

Table: $\left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$

$\left(\frac{1}{2}\right)^{n-1} u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \quad \left(\begin{matrix}\text{Time}\\\text{shift}\end{matrix}\right)$

$$x[n] = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

Table: $Y(z) = \dfrac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}, \quad |z| > 1/2$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}\left[\frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}\right]^{-1}$$

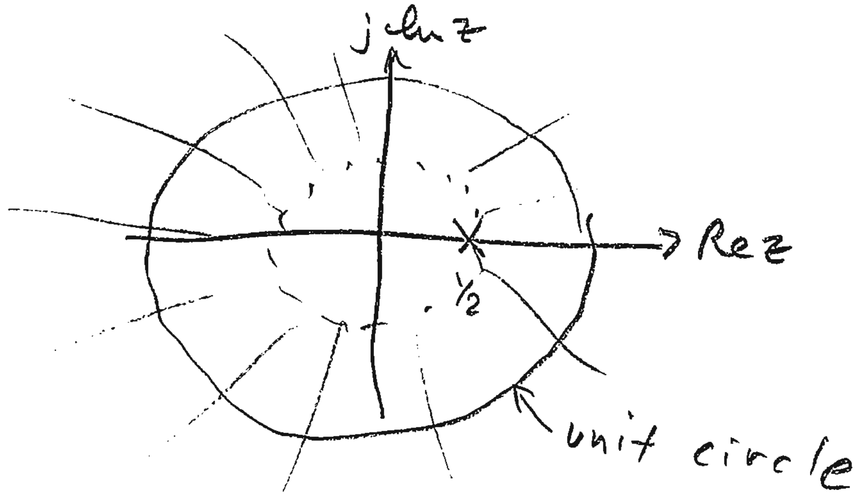
$$= \frac{\frac{1}{2}z^{-1}\left(1-\frac{1}{2}z^{-1}\right)}{\frac{1}{2}z^{-1}\left(1-\frac{1}{2}z^{-1}\right)^2} = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$\boxed{H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}}$$

4-3

b) is H stable?

YES. The ROC of $H(z)$ includes the unit circle.



c) Find $f[n]$.

$$g[n] = \frac{3}{2}\delta[n-1] \overset{Z}{\longleftrightarrow} G(z) = \frac{3}{2}z^{-1}, \quad |z|>0$$

$$H(z) = \frac{F(z)}{1+F(z)G(z)} \Rightarrow \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{F(z)}{1+\frac{3}{2}z^{-1}F(z)}$$

cross multiply: $1+\frac{3}{2}z^{-1}F(z) = \left[1-\frac{1}{2}z^{-1}\right]F(z)$

$$1+\frac{3}{2}z^{-1}F(z) = F(z) - \frac{1}{2}z^{-1}F(z)$$

$\longrightarrow$

c)...

$$1 = F(z) - \tfrac{3}{2}z^{-1}F(z) - \tfrac{1}{2}z^{-1}F(z)$$

$$1 = F(z) - 2z^{-1}F(z)$$

$$1 = F(z)\left[1 - 2z^{-1}\right]$$

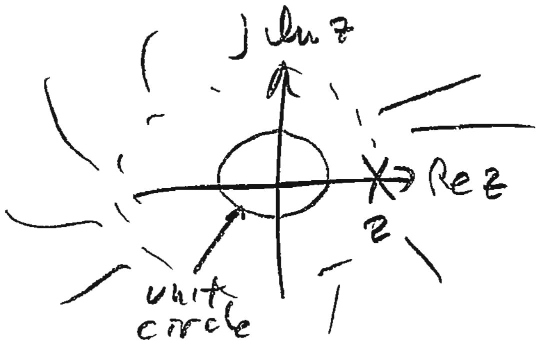
$$F(z) = \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

(since F is causal as we determined in part (a), f[n] must be right-sided and the ROC of F(z) must be exterior).

Table: $\boxed{f[n] = 2^n u[n]}$

d) is F stable?

<u>**No**</u>, The ROC of F(z) does not include the unit circle.



ROC is <u>$|z| > 2$</u>

(5) H is an LTI system with I/O equation

$$y[n] + \frac{5}{2}y[n-1] - \frac{3}{2}y[n-2] = x[n] - 4x[n-1]$$

a) Find $H(z)$ and give a pole/zero plot:

- Take z-transform on both sides, applying the time shift property:

$$Y(z) + \frac{5}{2}z^{-1}Y(z) - \frac{3}{2}z^{-2}Y(z) = X(z) - 4z^{-1}X(z)$$

$$Y(z)\left[1 + \frac{5}{2}z^{-1} - \frac{3}{2}z^{-2}\right] = X(z)\left[1 - 4z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-4z^{-1}}{1+\frac{5}{2}z^{-1} - \frac{3}{2}z^{-2}}$$

$$= \frac{1-4z^{-1}}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}$$

- There is a zero at $z=4$
- There are poles at $z=\frac{1}{2}$ and $z=-3$.
- We must also check the points $z=0$ and $z=\infty$:

$$\lim_{z\to 0} H(z) = \lim_{z\to 0} \frac{1-4\frac{1}{z}}{(1-\frac{1}{2}\frac{1}{z})(1+3\frac{1}{z})} \longrightarrow \frac{\infty}{-\infty^2} \longrightarrow 0$$

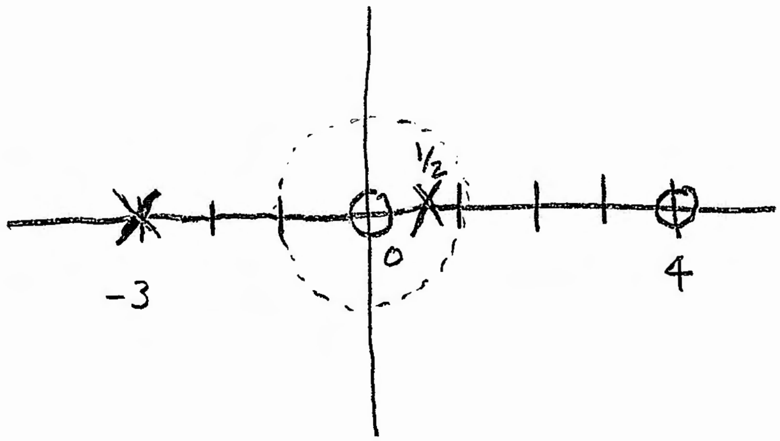
$\Rightarrow H(z)$ has a <u>zero</u> at $z=0$

$$\lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{1-4\frac{1}{z}}{(1-\frac{1}{2}\frac{1}{z})(1+3\frac{1}{z})} = \frac{1-4\cdot 0}{(1-\frac{1}{2}\cdot 0)(1+3\cdot 0)} = \frac{1}{1} = 1$$

$\Rightarrow H(z)$ does <u>not</u> have a pole at $z=\infty$ and does <u>not</u> have a zero at $z=\infty$.

So, the zeros of $H(z)$ are at $z=4$ and $z=0$.
the poles are at $z=\frac{1}{2}$ and $z=-3$.

P/Z plot:



b) If H is stable, then the ROC of $H(z)$ must include the unit circle. So the ROC must be $\frac{1}{2} < |z| < 3$.

– To find $h[n]$, we have to do a PFE on $H(z)$:

$$H(z) = \frac{1-4z^{-1}}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+3z^{-1}}$$

$$A = \left.\frac{1-4\theta}{1+3\theta}\right|_{\theta=2} = \frac{1-8}{1+6} = \frac{-7}{7} = -1$$

$$B = \left.\frac{1-4\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-\frac{1}{3}} = \frac{1+\frac{4}{3}}{1+\frac{1}{6}} = \frac{7/3}{7/6} = \frac{7}{3}\cdot\frac{6}{7} = \frac{6}{3} = 2$$

$\longrightarrow$

So $$H(z) = \underbrace{\frac{-1}{1-\frac{1}{2}z^{-1}}}_{\substack{\text{ROC for}\\ \text{this term}\\ \text{must be}\\ |z| > \frac{1}{2}}} + \underbrace{\frac{2}{1+3z^{-1}}}_{\substack{\text{ROC for this}\\ \text{term must}\\ \text{be } |z| < 3}}$$

$$\frac{-1}{1-\frac{1}{2}z^{-1}},\ |z| > \frac{1}{2} \xleftrightarrow{\text{Table}} -\left(\tfrac{1}{2}\right)^n u[n]$$

$$\frac{2}{1+3z^{-1}},\ |z| < 3 \xleftrightarrow{\text{Table}} -2(-3)^n u[-n-1]$$

$$\boxed{h[n] = -\left(\tfrac{1}{2}\right)^n u[n] - 2(-3)^n u[-n-1]}$$

c) Now assume that H is <u>causal</u>. For H to be causal, the ROC must be <u>exterior</u> to the largest pole.

ROC: $|z| > 3$

$\longrightarrow$

- The PFE is the same as in part (b), but the ROC is different.

$$H(z) = \underbrace{\frac{-1}{1-\frac{1}{2}z^{-1}}}_{\substack{\text{ROC for this} \\ \text{term must} \\ \text{be } |z|>\frac{1}{2}}} + \underbrace{\frac{2}{1+3z^{-1}}}_{\substack{\text{ROC for this} \\ \text{term must} \\ \text{be } |z|>3}}$$

$$\frac{-1}{1-\frac{1}{2}z^{-1}}, \quad |z|>\frac{1}{2} \quad \xleftrightarrow{\text{Table}} \quad -\left(\frac{1}{2}\right)^n u[n]$$

$$\frac{2}{1+3z^{-1}}, \quad |z|>3 \quad \xleftrightarrow{\text{Table}} \quad 2(-3)^n u[n]$$

$$\boxed{h[n] = -\left(\frac{1}{2}\right)^n u[n] + 2(-3)^n u[n]}$$

d) For H to be causal, the ROC must be exterior to the largest pole. It must be $|z|>3$.

For H to be stable, the ROC must include the unit circle where $|z|=1$.

$\Rightarrow$ These cannot both be true.

- If the system is causal, then the ROC is $|z|>3$, which cannot include the unit circle

$\longrightarrow$

- If the system is stable, then the ROC must include the unit circle, so it cannot be exterior to the largest pole.

$\Rightarrow$ Therefore, for this difference equation, there is <u>no</u> system H that is both causal and stable.