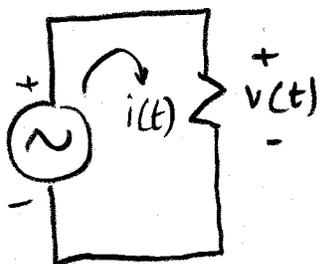


- A signal is a manifestation or realization of a physical quantity

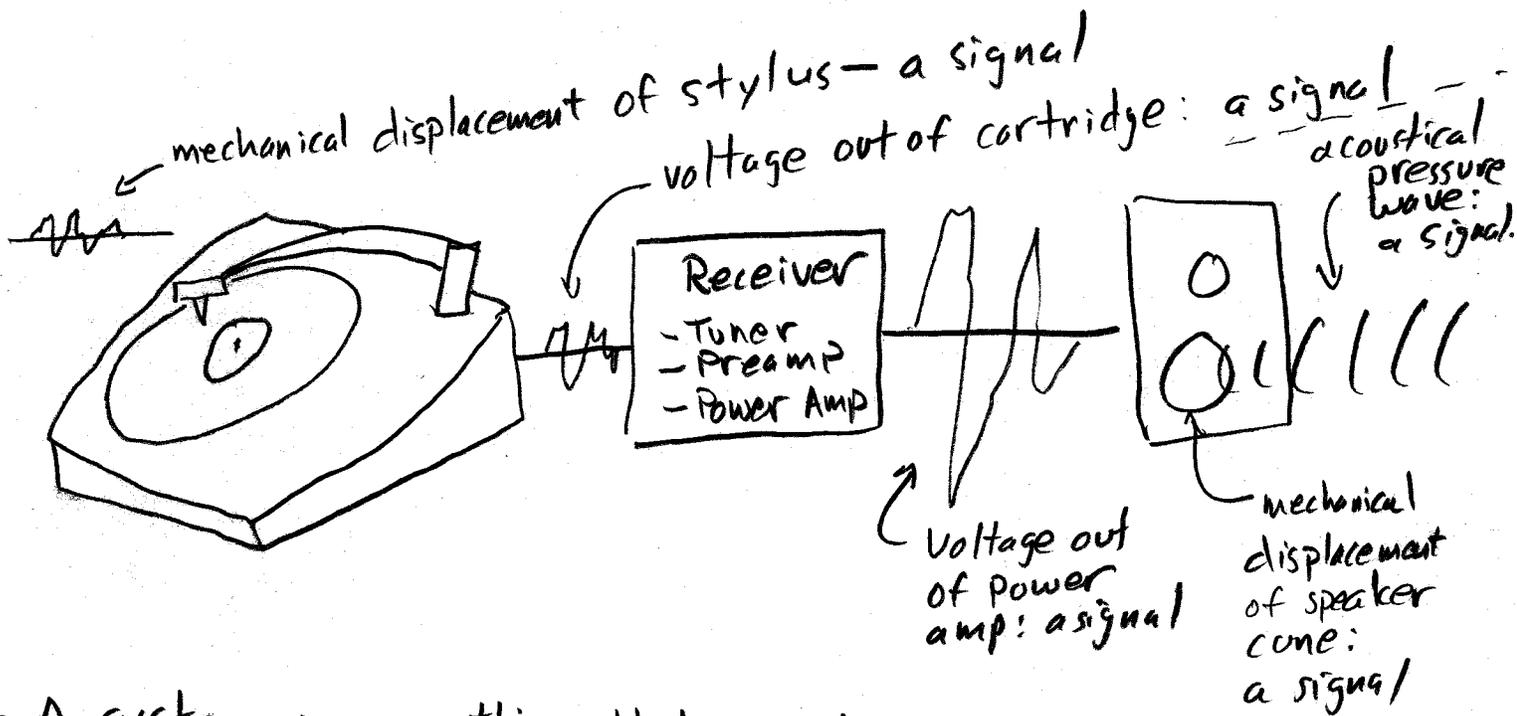
- You can think of a signal as a measurement of one physical quantity with respect to another.

EX:



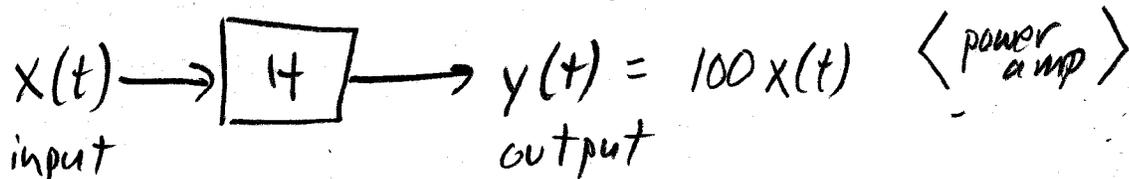
- current $i(t)$ is a signal

- voltage $v(t)$ is a signal

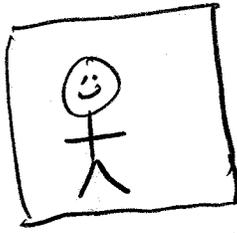


- A system is anything that inputs one (or more) signal(s) and outputs one (or more) signal(s).

⇒ The system usually transforms the input into the output in a predictable way.



- Another example of a signal: an image $I(x,y)$



- Now time is not one of the independent variables.

- But space (horiz, vert) is!

- For engineering analysis & design, we model signals & systems with mathematical objects,

⇒ Then we can use the tools of mathematics for analysis & design.

- The mathematical objects we use to model signals are functions.

⇒ for some signals like the Dirac delta $\delta(t)$ we will have to use a more powerful math object called a distribution.
More on this later, we won't go into too much detail.

- The mathematical objects we use to model systems are operators.

⇒ operators are "function functions"! They gobble up one function and output another one.

DEF: if the members of a set can be matched up with integers, then the set is called countable.

- Intuitively, this means that you can "count" the members.
- All finite sets are countable.
- Infinite sets are countable if the number of members is not more than the number of integers.
- The members of a countable set can also be indexed by integers. So, for example, we can write a_k , where k stands for integers.
- For a countable set of numbers, we can add them up using a "capital sigma do loop":

$$\text{sum} = \sum_{k=-\infty}^{\infty} a_k$$

↑ the loop counter

Examples of countable sets:

- The domain "A" of the function on page 1.3.
- The natural numbers: $\mathbb{N} = 1, 2, 3, \dots$

An example of an uncountable set:

- The real numbers \mathbb{R} .

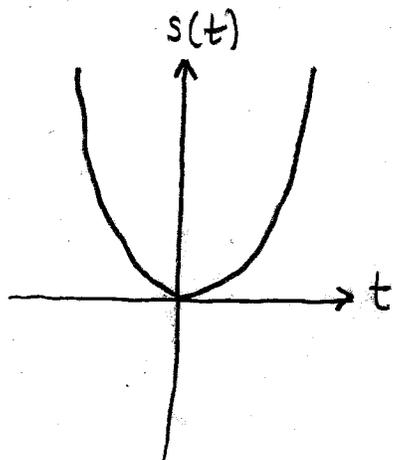
DEF: a function that has a countable domain is called a "discrete-domain" function or "discrete-time" function (if the domain represents time).

DEF: a function that has an uncountable domain is called a "continuous-domain" function or "continuous-time" function (if the domain represents time).

- The function on page 1.3 is a discrete-domain function, because the domain A is countable.

- Another example function: $s(t) = t^2$

Domain: \mathbb{R} , the set of real numbers } $s: \mathbb{R} \rightarrow \mathbb{R}$
Range: \mathbb{R} , the set of real numbers }



- when we write $s(t) = t^2$, we mean that "t" stands for a member of the domain.

- In symbols: $t \in \mathbb{R}$.
read "t is in \mathbb{R} ".

- In this case, the domain is uncountable, so s is a continuous-time function.

NOTE: - All real numbers in the domain get mapped by the function s .
- Not all real numbers in the range get mapped to (-4 for example) PAGE 1.5

- The members of an uncountable set can't be matched up with integers.

- So they also can't be indexed by integers.

⇒ For an uncountable set of numbers, you cannot add them up with a "capital sigma do loop."

- But, in a certain sense, you can add them up using "capital Roman S addition"...

e.g., integration:

$$\text{"sum"} = \int_a^b s(t) dt$$

So: to add up numbers from a countable set, use a "capital sigma do loop":

$$\text{sum} = \sum_{k=a}^b x[k]$$

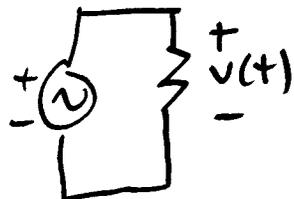
- to add up numbers from an uncountable set, use "capital Roman S addition" (integration):

$$\text{"Sum"} = \int_a^b x(t) dt$$

- A continuous-time signal has domain \mathbb{R} . The signal, e.g., has a value at every time.

\Rightarrow For a continuous-time signal, you cannot count the number of places where the signal has a value.

EX:

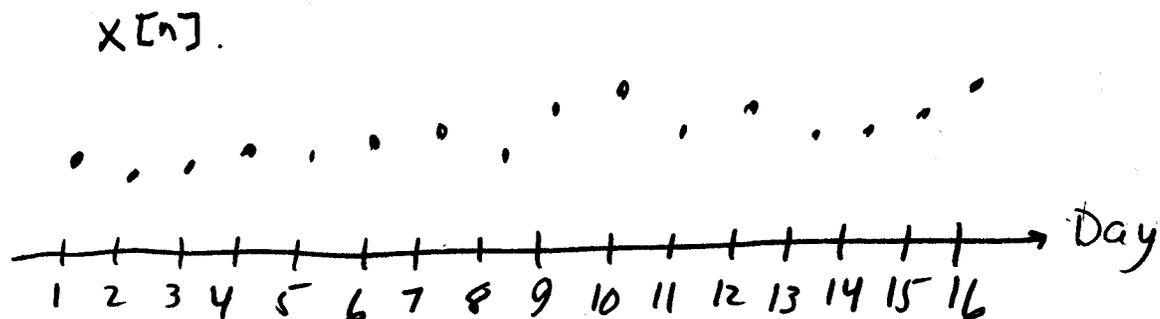


- The voltage $v(t)$ has a value at every time. It is a continuous-time signal.

- A discrete-time signal has domain \mathbb{Z} , the integers.

You can count the number of places where the signal has a value.

EX: The closing price each day of some particular stock.



\rightarrow Some signals like the one above are inherently discrete-time.

\rightarrow You can also make a discrete time signal by sampling a continuous-time signal at a countable number of times.

EX: digital audio compact disk (CD).

MAKING A DISCRETE-TIME SIGNAL FROM A CONTINUOUS-TIME SIGNAL:



- You have already seen continuous-time signals and systems in ECE 2723.
- You have already seen discrete-time signals and systems in ECE 2713.

Important Sets

\mathbb{R} the set of real numbers

$\mathbb{R} \times \mathbb{R}$ or \mathbb{R}^2 : The set of ordered pairs $\begin{bmatrix} x \\ y \end{bmatrix}$ where x is in \mathbb{R} and y is in \mathbb{R} .

\mathbb{R}^n the set of ordered n -tuples $[x_1, x_2, \dots, x_n]^T$ where each x_i is in \mathbb{R} .

\mathbb{C} the set of complex numbers. For each z in \mathbb{C} , we can write $z = a + jb$ where a and b are in \mathbb{R} . So, geometrically, \mathbb{C} is identical to \mathbb{R}^2 .

\mathbb{Z} the set of integers

\mathbb{Q} the set of rationals. Each rational can be written as p/q where p and q are in \mathbb{Z} .

\mathbb{N} The set of natural or counting numbers $1, 2, 3, 4, \dots$

Symbols

\forall	"for all"
\in	"in" or "is an element of"
\exists	"there exists"
s.t.	"such that"

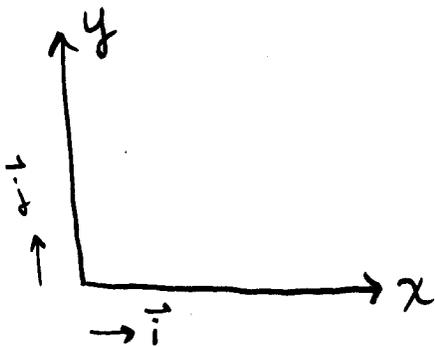
EX:

$\forall x \in \mathbb{R}$ "for all x in the reals"

$\exists x \in \mathbb{R}$ s.t. $x > 5$ "there exists an x in the reals such that x is greater than 5"
(6, for example)

Signal Representations

- First, we review some linear algebra on \mathbb{R}^2 .



$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- \mathbb{R}^2 is the set of ordered pairs (x, y) or $\begin{bmatrix} x \\ y \end{bmatrix}$.
- Each such ordered pair describes a point in the plane.

- The set of points, or vectors, $\{\vec{i}, \vec{j}\}$ is an orthonormal basis for \mathbb{R}^2 .

- This means:

1. Any vector $\vec{v} \in \mathbb{R}^2$ can be written as a linear combination of the basis vectors:

$$\vec{v} = c_1 \vec{i} + c_2 \vec{j}$$

where c_1 and c_2 are constants.

2. Each basis vector has unit length.

3. All of the vectors in the basis are mutually orthogonal:

$$\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

DEF:

- Given a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ and a vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$, the dot product is defined by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 = \sum_{n=1}^2 x_n y_n$$

EX: $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\vec{x} \cdot \vec{y} = 1 \cdot 3 + 2 \cdot 4 = 3 + 8 = \underline{\underline{11}}$$

← a number, not a vector.

NOTES

- In general, the dot product is written using angle brackets:

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y}$$

- If one or both of the vectors have complex-valued entries, then you must conjugate the entries of the second vector when computing the dot product:

$$\vec{x}, \vec{y} \in \mathbb{C}^n,$$

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_i^*$$

Question: given a vector $\vec{x} \in \mathbb{R}^2$, how do we write \vec{x} as a linear combination of the basis $\{\vec{i}, \vec{j}\}$?

Answer: Use the dot product.

$$\vec{x} = c_1 \vec{i} + c_2 \vec{j}$$

$$\text{where } c_1 = \langle \vec{x}, \vec{i} \rangle$$

$$c_2 = \langle \vec{x}, \vec{j} \rangle$$

In other words,

$$\vec{x} = \langle \vec{x}, \vec{i} \rangle \vec{i} + \langle \vec{x}, \vec{j} \rangle \vec{j}.$$

EX: $\vec{x} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$$\vec{x} = \langle \vec{x}, \vec{i} \rangle \vec{i} + \langle \vec{x}, \vec{j} \rangle \vec{j}$$

$$= \langle \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \langle \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= (5 \cdot 1 + 8 \cdot 0) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (5 \cdot 0 + 8 \cdot 1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 8 \end{bmatrix} \checkmark$$

- This may seem overly simple, but the approach works in general.

$$\text{EX: } \vec{e}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

is also an orthonormal basis for \mathbb{R}^2 .

- To write a vector $\vec{x} \in \mathbb{R}^2$ as a linear combination of this basis, use the same strategy:

$$\vec{x} = \langle \vec{x}, \vec{e}_1 \rangle \vec{e}_1 + \langle \vec{x}, \vec{e}_2 \rangle \vec{e}_2.$$

- The space \mathbb{R}^3 is the set of ordered triples $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1, x_2, x_3 \in \mathbb{R}$.

- One orthonormal basis for \mathbb{R}^3 is the set $\{\vec{i}, \vec{j}, \vec{k}\}$ where

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- The dot product in \mathbb{R}^3 is defined just like you expect.

For $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, the dot product is

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 \\ &= \sum_{i=1}^3 x_i y_i \end{aligned}$$

- To write a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ as a linear combination of the basis $\{\vec{i}, \vec{j}, \vec{k}\}$, proceed as before:

$$\vec{x} = \underbrace{\langle \vec{x}, \vec{i} \rangle}_{c_1} \vec{i} + \underbrace{\langle \vec{x}, \vec{j} \rangle}_{c_2} \vec{j} + \underbrace{\langle \vec{x}, \vec{k} \rangle}_{c_3} \vec{k} \quad (\text{do an example})$$

⇒ Don't forget to conjugate the entries of the second vector if they are complex-valued.

- The procedure works in higher dimensional spaces as well.

- Consider \mathbb{R}^{3793} .

- A vector $\vec{x} \in \mathbb{R}^{3793}$ looks like

$$\vec{x} = [x_1 \ x_2 \ x_3 \ \dots \ x_{3793}]^T$$

- If $\vec{x}, \vec{y} \in \mathbb{R}^{3793}$, then the dot product is

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{3793} x_i y_i$$

⇒ Don't forget to conjugate if the vectors have complex entries.

- if the set of vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \dots, \vec{e}_{3793}\}$ is an orthonormal basis for \mathbb{R}^{3793} , then any vector $\vec{x} \in \mathbb{R}^{3793}$ can be written as

$$\begin{aligned} \vec{x} &= \langle \vec{x}, \vec{e}_1 \rangle \vec{e}_1 + \langle \vec{x}, \vec{e}_2 \rangle \vec{e}_2 + \dots + \langle \vec{x}, \vec{e}_{3793} \rangle \vec{e}_{3793} \\ &= \sum_{n=1}^{3793} \langle \vec{x}, \vec{e}_n \rangle \vec{e}_n. \end{aligned}$$

- This idea extends easily to infinite dimensional vector spaces too.

- Consider vectors of the form

$$\vec{x} = [\dots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \dots]^T$$

where the index runs over all the integers from $-\infty$ to ∞ .

- This kind of vector is identical to a discrete-time signal $x[n]$ of the type you studied in ECE 2713.

- For two such signals \vec{x}, \vec{y} (or $x[n], y[n]$), the dot product is just

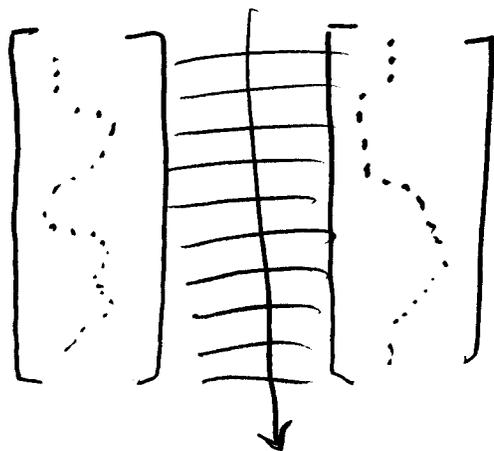
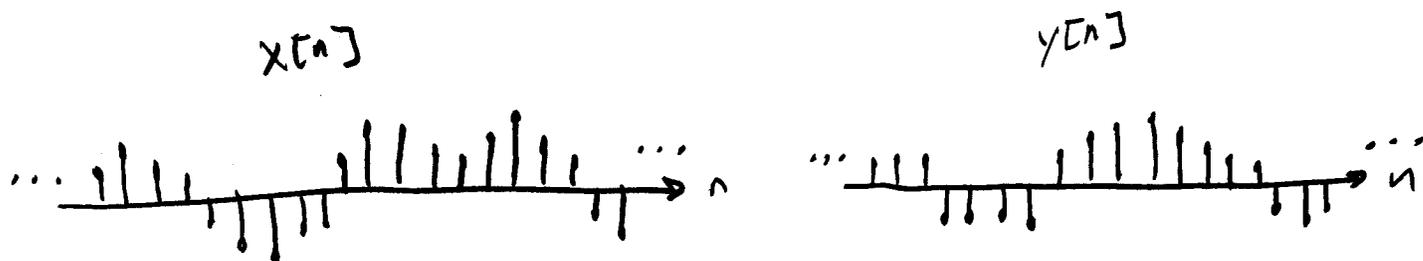
$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=-\infty}^{\infty} x_i y_i^* = \sum_{i=-\infty}^{\infty} x[i] y[i]^*$$

- The idea of "dot product" for two signals $x[n]$ and $y[n]$ is:

1. Load the signal values into vectors
2. Stand the vectors up beside each other
3. If the entries are complex, conjugate the second vector
4. Multiply corresponding entries
5. Add up the products.

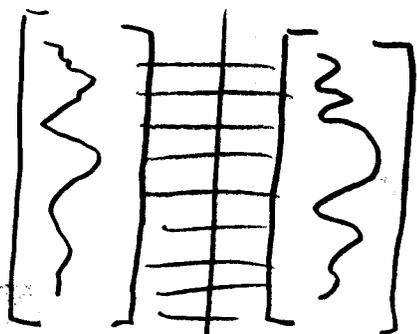
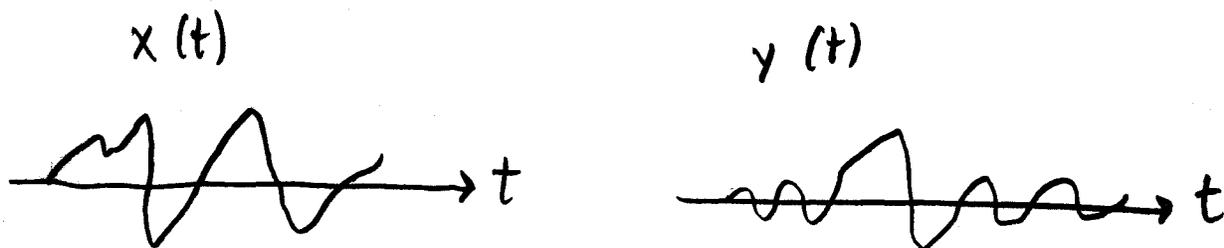


- In pictures:



a number, $\langle x[n], y[n] \rangle = \sum_{i=-\infty}^{\infty} x[i] y[i]^*$

- The concept extends to continuous-time signals as well:

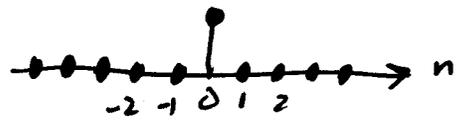


$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

So what?

- Recall the discrete-time signal (Kronecker delta)

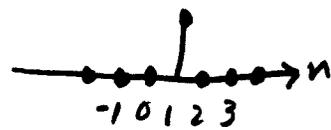
$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{other} \end{cases}$$



- The signal is "turned on" at $n=0$.

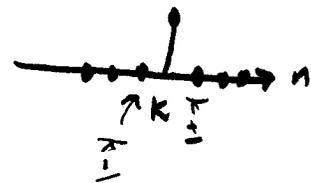
- write it as a vector: $[\dots 0 0 \underset{\substack{\uparrow \\ n=0}}{1} 0 0 \dots]^T$

- The signal $\delta[n-1] = \begin{cases} 1 & n=1 \\ 0 & \text{other} \end{cases}$



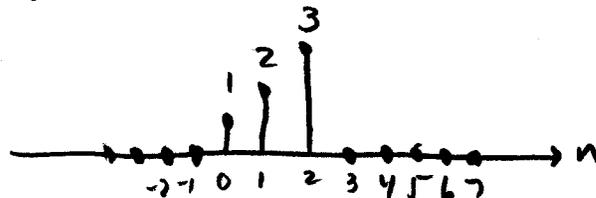
- The signal $\delta[n-k] = \begin{cases} 1, & n=k \\ 0 & \text{other} \end{cases}$

is turned on at $n=k$.



\Rightarrow So, for discrete-time signals $x[n]$, the set $\{\delta[n-k]\}_{k \in \mathbb{Z}}$ plays the same role as the set $\{\vec{i}, \vec{j}, \vec{k}\}$ plays in \mathbb{R}^3 .

- Consider the signal $x[n]$



- Obviously, $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$

$$= \underset{\substack{\uparrow \\ 0}}{1} + \underset{\substack{\uparrow \\ 1}}{2} + \underset{\substack{\uparrow \\ 2}}{3}$$

- In ECE 3793, we will often need to write a signal in terms of more than one basis.

- Why?

- just like changing coordinates (say, rectangular to spherical) sometimes makes a calculus problem easier,

- changing basis will sometimes make a problem easier in ECE 3793.

- Recall: The set $\{\delta[n-k]\}_{k \in \mathbb{Z}}$ is, as we have seen, an orthonormal basis for the set of discrete-time signals $x[n]$.

- FACT: The set $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi]}$ is also a basis.

Note: $e^{j\omega n} = \cos \omega n + j \sin \omega n$

→ There is one basis "vector" for each $\omega \in [-\pi, \pi]$

→ The basis is not orthonormal. Each basis vector has length $\sqrt{2\pi}$ (not one!).

→ But this basis is orthogonal.

- Given a signal $x[n]$, let's write $x[n]$ as a linear combination of the basis $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi]}$

Step 1: Use the dot product to find the required coefficients:

For any particular value $\omega \in [-\pi, \pi]$, we have the basis "signal" $e^{j\omega n}$ with required coefficient c_ω given by

$$\begin{aligned}c_\omega &= \langle x[n], e^{j\omega n} \rangle \\&= \sum_{n=-\infty}^{\infty} x[n] (e^{j\omega n})^* \\&= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}\end{aligned}$$

\Rightarrow This is called the discrete-time Fourier transform.

\rightarrow Usually, we write all of the coefficients for all of the basis signals together as a function of " ω " (indexing set) using the notation $X(e^{j\omega})$:

$$X(e^{j\omega}) = \langle x[n], e^{j\omega n} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Step 2: Add up all the coefficients times the corresponding basis signals to get our signal $x[n]$.

Note: Since the basis signals have length $\sqrt{2\pi}$ instead of 1, we have to scale the coefficients by $\frac{1}{2\pi}$.

$$\begin{aligned}x[n] &= \frac{1}{2\pi} \text{Add}_{\omega \in [-\pi, \pi]} \left\{ (\text{coef}) (\text{basis signal}) \right\} \\&= \frac{1}{2\pi} \text{Add}_{\omega \in [-\pi, \pi]} \left\{ c_{\omega} e^{j\omega n} \right\} \\&= \frac{1}{2\pi} \text{Add}_{\omega \in [-\pi, \pi]} \left\{ x(e^{j\omega}) e^{j\omega n} \right\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} x(e^{j\omega}) e^{j\omega n} d\omega\end{aligned}$$

— This is called the discrete-time inverse Fourier transform.

- Now consider the set of continuous-time signals $x(t)$.

FACT: The set $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$ is a basis for this set of signals,

- As in discrete time, this basis is not orthonormal.
- Each basis signal has length $\sqrt{2\pi}$.
- So, if we use the dot product to write $x(t)$ as a linear combination of the basis $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$, we will have to scale the coefficients by $\frac{1}{2\pi}$.

Step 1: for each basis signal, use the dot product to find the required coefficient:

$$\begin{aligned} c_\omega &= \langle x(t), e^{j\omega t} \rangle \\ &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &\equiv X(\omega) \end{aligned}$$

This is called the Fourier transform of $x(t)$.

Step 2 : Add up all the coefficients times the corresponding basis signals to get our signal $x(t)$:

(don't forget to scale by $\frac{1}{2\pi}$!)

$$x(t) = \frac{1}{2\pi} \text{Add}_{\omega \in \mathbb{R}} \{ (\text{coef})(\text{basis signal}) \}$$

$$= \frac{1}{2\pi} \text{Add}_{\omega \in \mathbb{R}} \{ c_{\omega} e^{j\omega t} \}$$

$$= \frac{1}{2\pi} \text{Add}_{\omega \in \mathbb{R}} \{ X(\omega) e^{j\omega t} \}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

- This is called the inverse Fourier transform.

- Now suppose we have an $x(t)$ who is a "bad guy."

→ as $t \rightarrow \infty$, $x(t)$ grows and grows out of control:



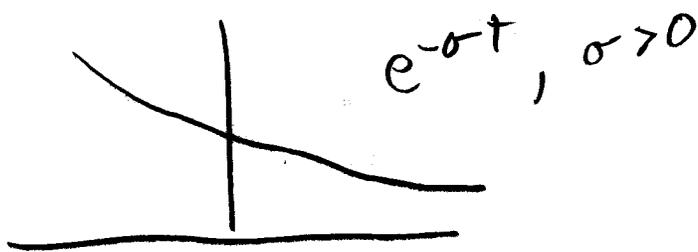
- We want to write $x(t)$ as a linear combination of the basis $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$.

- We try to compute the coefficients $X(\omega)$ using the dot product:

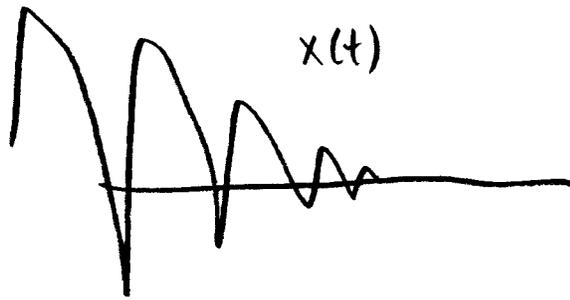
$$X(\omega) = \langle x(t), e^{j\omega t} \rangle = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- But this integral blows up, because $x(t)$ is so bad.

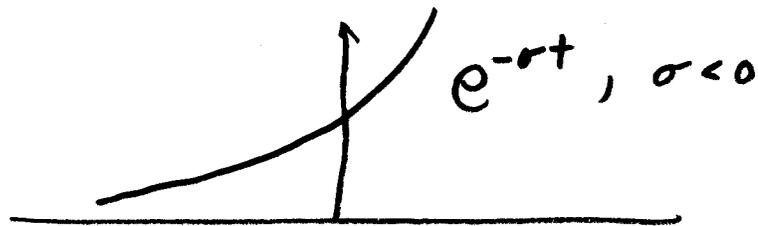
- Maybe we can fix things up if we multiply the signal by a decaying exponential $e^{-\sigma t}$, $\sigma \in \mathbb{R}$, $\sigma > 0$:



- For a signal that is "bad" on the other side,



we may have to use a "growing" exponential $e^{-\sigma t}$ with $\sigma < 0$:



- So in general, for "bad guys" $x(t)$ we will write the fixed up signal $e^{-\sigma t} x(t)$ as a linear combination of the basis $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$.

- The "rate" σ of the exponential will be a parameter.

- We will call the required coefficients (dot products)

$X(\sigma, \omega)$:

$$X(\sigma, \omega) = \langle e^{-\sigma t} x(t), e^{j\omega t} \rangle$$

$$= \int_{-\infty}^{\infty} e^{-\sigma t} x(t) e^{-j\omega t} dt$$

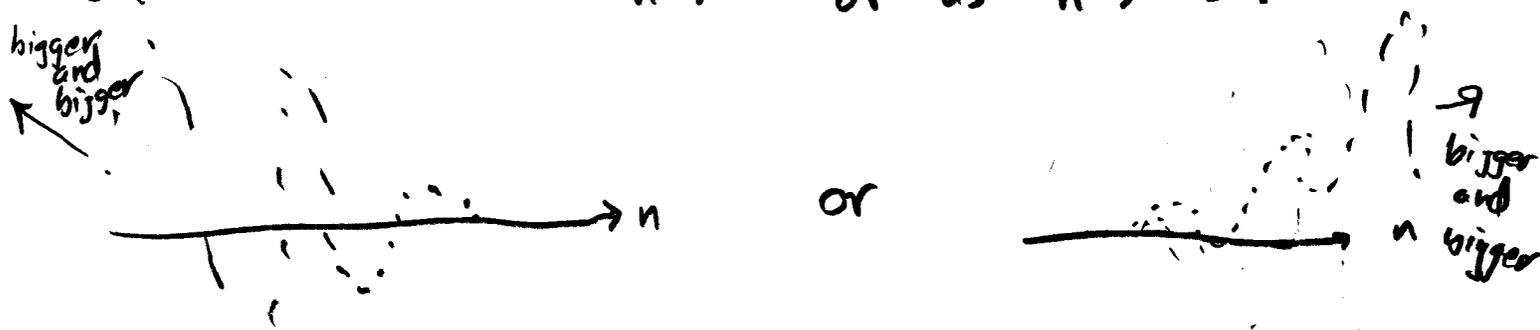
$$= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

- By defining a complex variable $s = \sigma + j\omega$, we can write the σ and $j\omega$ together to get

$$\begin{aligned} X(s) &= \langle e^{-\sigma t} x(t), e^{j\omega t} \rangle \\ &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \end{aligned}$$

- This is called the Laplace transform of $x(t)$.
- The procedure for adding up the coefficients times the basis functions to get back to $x(t)$ is called the inverse Laplace transform.
- It is similar to what we have already done with the Fourier and discrete-time Fourier transforms, but a little more complicated because of the $e^{-\sigma t}$ "fixer upper" function.
- We will cover it in detail in chapter 9.

- Now suppose we have a "bad guy" $x[n]$ who grows out of control as $n \rightarrow \infty$ or as $n \rightarrow -\infty$.



- When we try to use the dot product to write $x[n]$ as a linear combination of the basis $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi]}$, the coefficients

$$\begin{aligned} X(e^{j\omega}) &= \langle x[n], e^{j\omega n} \rangle \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \end{aligned}$$

blow up.

- Like before, we can fix this up by multiplying $x[n]$ times an appropriate exponential α^{-n} , $\alpha \in \mathbb{R}$, $\alpha > 0$, $n \in \mathbb{Z}$

→ if $\alpha > 1$, then α^{-n} is a decaying exponential:



→ if $0 < \alpha < 1$, then α^{-n} is a growing exponential:



- Now we will write the fixed up signal $\alpha^{-n} x[n]$ as a linear combination of the basis $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi]}$ using the dot product,

- The required coefficients are

$$\begin{aligned} X(\alpha, e^{j\omega}) &= \langle \alpha^{-n} x[n], e^{j\omega n} \rangle \\ &= \sum_{n=-\infty}^{\infty} \alpha^{-n} x[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n] \alpha^{-n} e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n] (\alpha e^{j\omega})^{-n} \end{aligned}$$

- But for any pair of reals α, ω , $\alpha e^{j\omega}$ is just a complex number in polar form.

- So define $z = \alpha e^{j\omega}$. Then we have

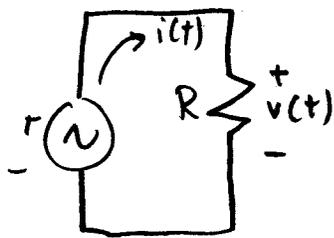
$$\begin{aligned} X(z) &= \langle \alpha^{-n} x[n], e^{j\omega n} \rangle \\ &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \end{aligned}$$

- This is called the Z-transform of $x[n]$.

- When we add up the coefficients $X(z)$ times the basis signals to get $x[n]$, it is called the inverse Z-transform.

- We will look at this in detail in Chapter 10.

Signal Energy & Power



- Voltage dropped on resistor is $v(t)$
- Current through resistor is $i(t)$
- Ohm's law: $v(t) = R i(t)$

- Instantaneous energy dissipated in resistor:

$$p(t) = v(t) i(t) = \frac{1}{R} v^2(t) = R i^2(t)$$

⇒ Inst. energy is proportional to the squares of the current and voltage signals.

→ It is customary in signal processing / control / computer to refer to squared signals as energy.

- The energy dissipated in the resistor from t_1 to t_2 is

$$\int_{t_1}^{t_2} \frac{1}{R} v^2(t) dt = \int_{t_1}^{t_2} R i^2(t) dt$$

- Power is energy per unit time.

- Average power dissipated in resistor from t_1 to t_2 is

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{R} v^2(t) dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} R i^2(t) dt$$

- Based on the above, engineers refer to the integral of any squared signal as "energy".

→ depending on the units of the signal, this "energy" might not have anything to do with actual energy in the "physics" sense. We call it energy anyway.

- So, for any signal $x(t)$, we define the energy to be

$$E_{\infty} = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

- For a discrete time signal $x[n]$, the energy is defined as

$$E_{\infty} = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

- These definitions are good for real-valued signals and for complex-valued signals. That's why we have the magnitudes $|x(t)|$ and $|x[n]|$ in the definitions.

- Likewise, we define the power of a continuous time signal $x(t)$ as

$$P_{\infty} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt,$$

→ even though this may have nothing to do with actual power in the physical sense.

- For a discrete time signal $x[n]$, the power is defined by

$$P_{\infty} = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2.$$

- For any signal,

→ if $E_{\infty} < \infty$, then $P_{\infty} = 0$.

→ if $0 < P_{\infty} < \infty$, then $E_{\infty} = \infty$.

- Note: There are some signals for which both E_{∞} and P_{∞} are infinite. Ex: $x(t) = e^t$.

- Signals for which $E_{\infty} < \infty$ are sometimes called "energy signals" or "finite energy signals".

- Signals for which $0 < P_{\infty} < \infty$ are sometimes called "power signals" or "finite power signals".

→ The energy of a power signal is infinite.

Note: The symbols E_{∞} and P_{∞} are not standard, but the book uses them, so we will too.

Transformations of the Independent Variable

① "Time Shifting" or "Translation" : $x(t-t_0)$

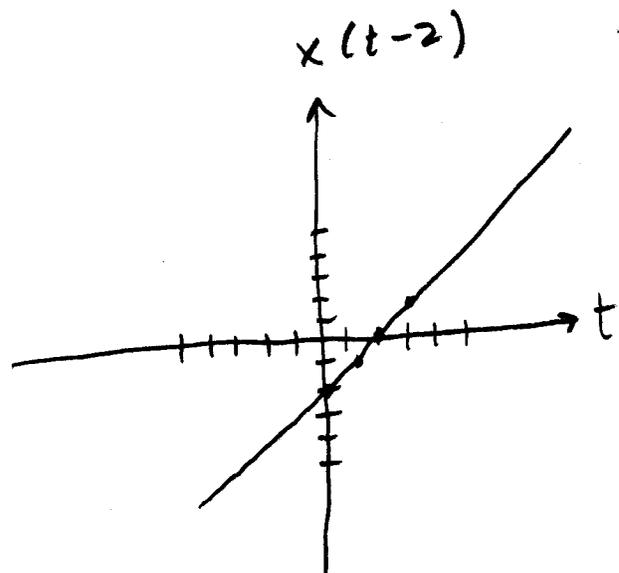
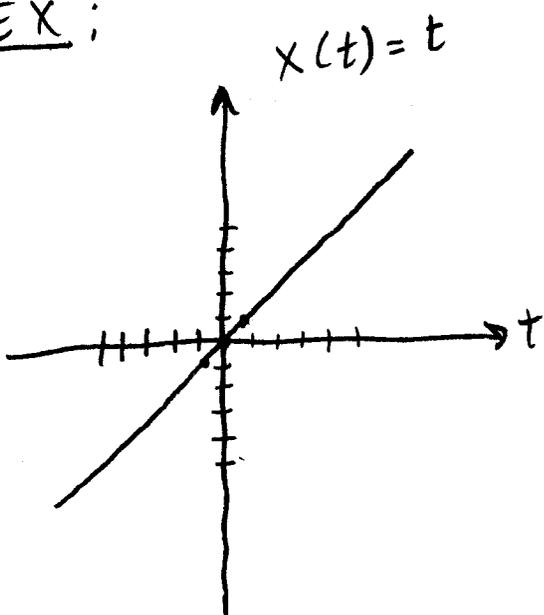
- If $t_0 > 0$, then the graph of $x(t-t_0)$ is the same as the graph of $x(t)$, but shifted right by t_0 .

why?

Suppose $t_0 = 2$. Then, say, when $t = 4$ $x(t)$ is doing $x(4)$. But $x(t-2)$ is doing $x(2)$, the same thing $x(t)$ did 2 seconds ago.

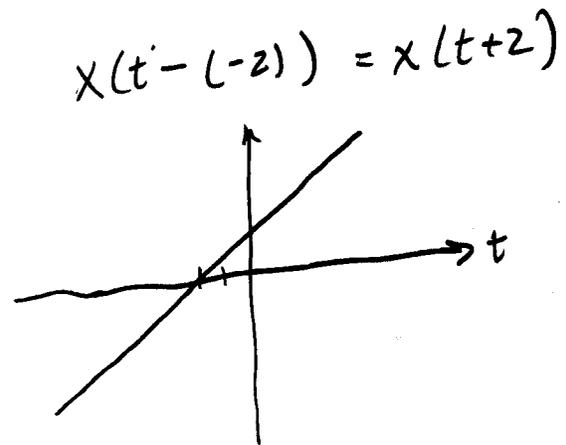
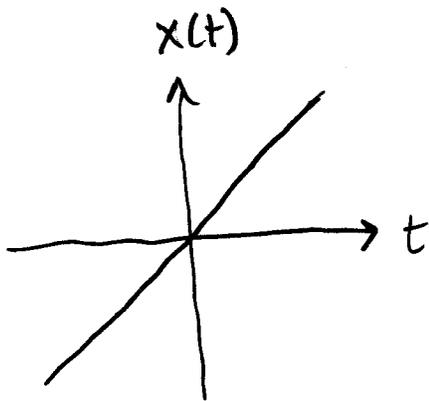
\Rightarrow So $x(t-2)$ does everything 2 seconds later than $x(t)$.

EX:



- if $t_0 < 0$, the graph of $x(t-t_0)$ is the same as $x(t)$ but shifted left by t_0 .

EX: $t_0 = -2$, $x(t) = t$



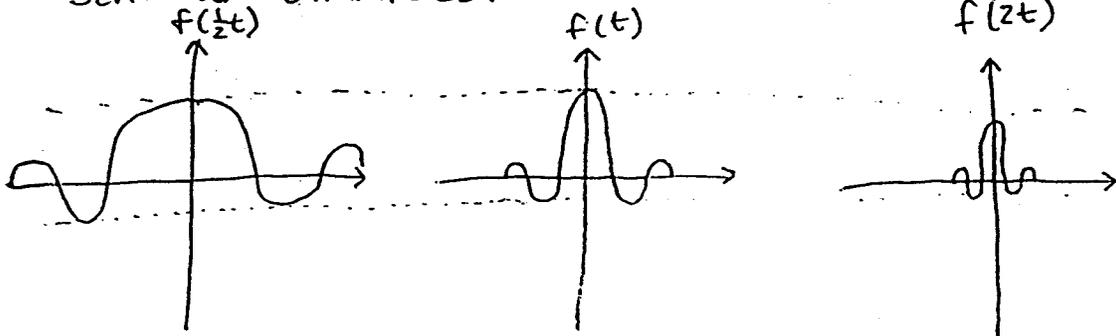
- For tests & HW, it will help to always do shifts the same way... always think of $x(t-t_0)$, whether t_0 is positive or negative.

② "Time Scaling" or "Dilation"; $x(at)$

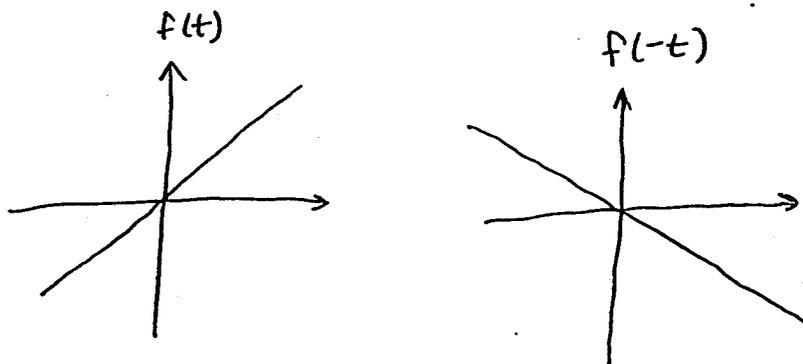
- if $a > 1$, the graph of $x(at)$ goes by faster than the graph of $x(t)$... so the graph gets squished.

- if $0 < a < 1$, the graph of $x(at)$ goes by slower than the graph of $x(t)$... so the graph gets stretched.

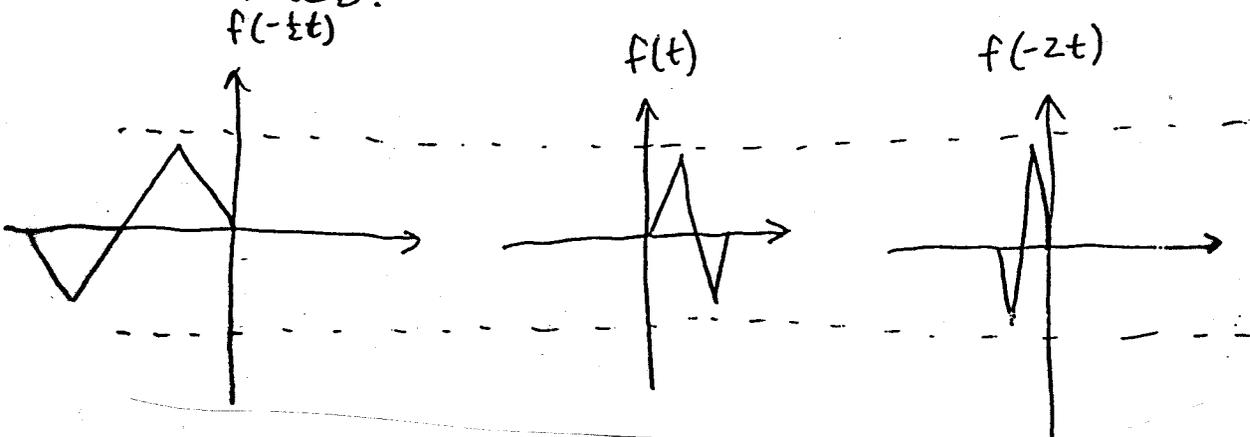
SCALING EXAMPLES:



- IN THE SPECIAL CASE WHERE $a = -1$, THIS IS CALLED REFLECTION BECAUSE IT FLIPS THE GRAPH OF THE FUNCTION AROUND THE VERTICAL AXIS: IF $f(t) = t$, THEN



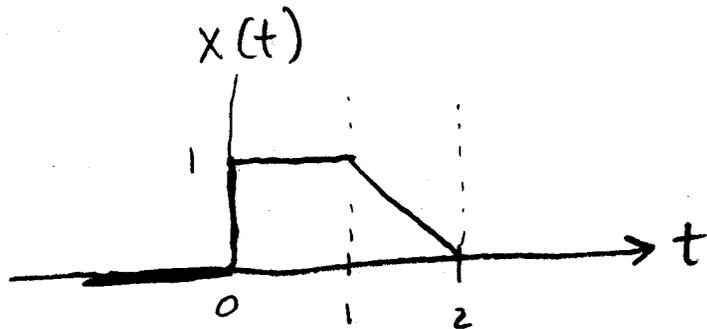
- GENERALLY, IF $a < 0$ THEN THE GRAPH OF THE FUNCTION IS REFLECTED AND SQUASHED OR STRETCHED:



③ Shift and Scale: $x(at-t_0)$

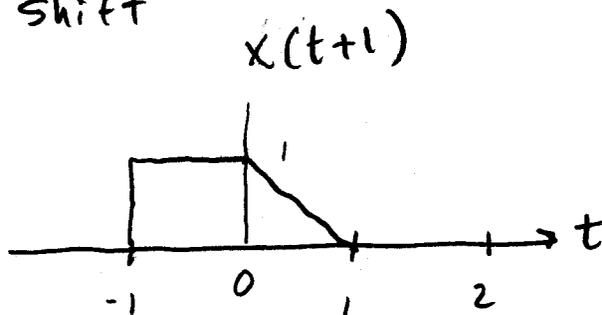
RULE OF THUMB: To obtain the graph of $x(at-t_0)$ from the graph of $x(t)$, do the shift first and the scale second.

EX: Book p. 10

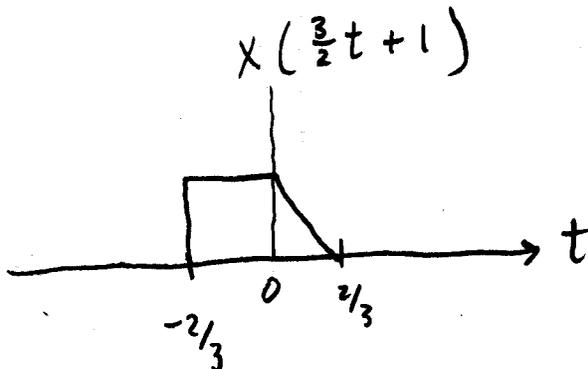


- graph $x(\frac{3}{2}t+1)$

Step 1: shift



Step 2: Scale

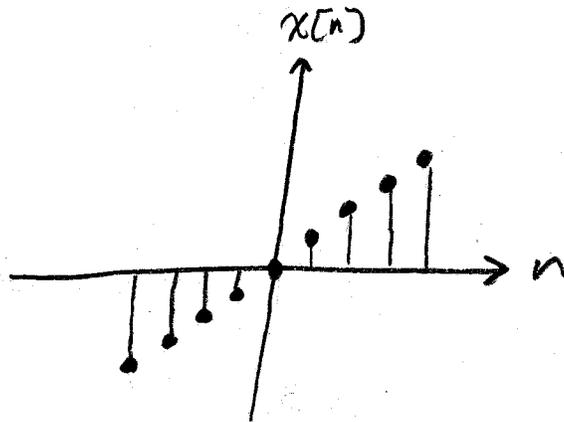


NOTE: $x[a(t-t_0)]$ is not the same as $x(at-t_0)$.

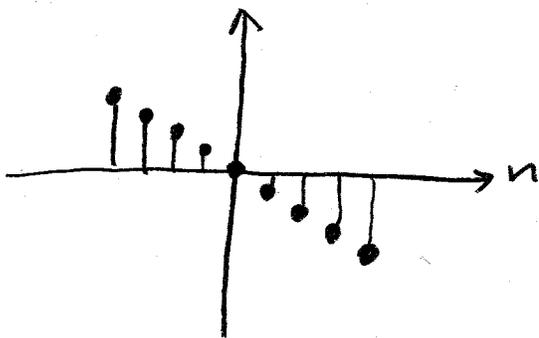
rather, $x[a(t-t_0)] = x(at-at_0)$. PAGE 1-35

Discrete Examples :

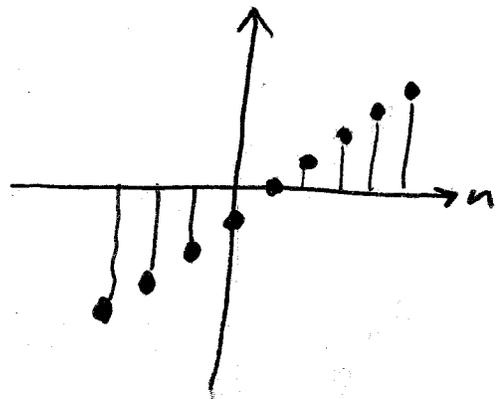
$$x[n] = n$$



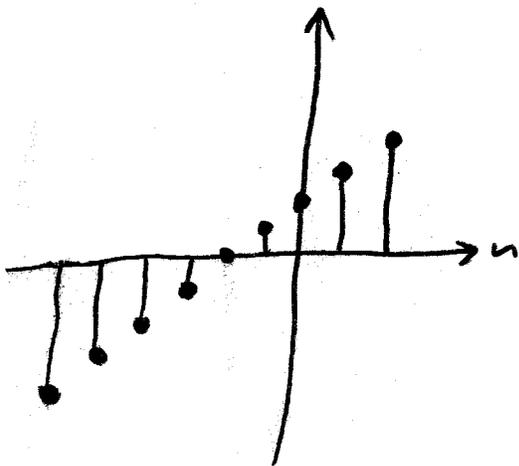
$$x[-n]$$



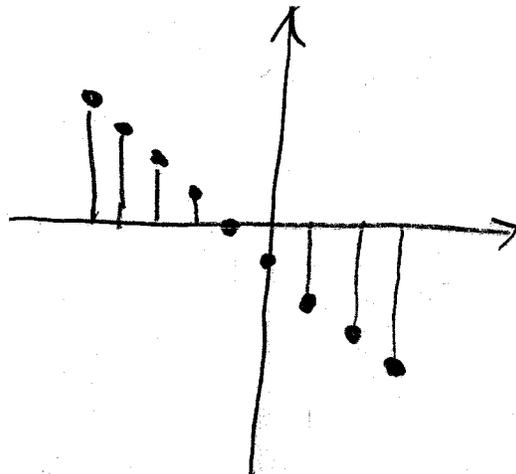
$$x[n-1]$$



$$x[n+2]$$



$$x[-n-1]$$



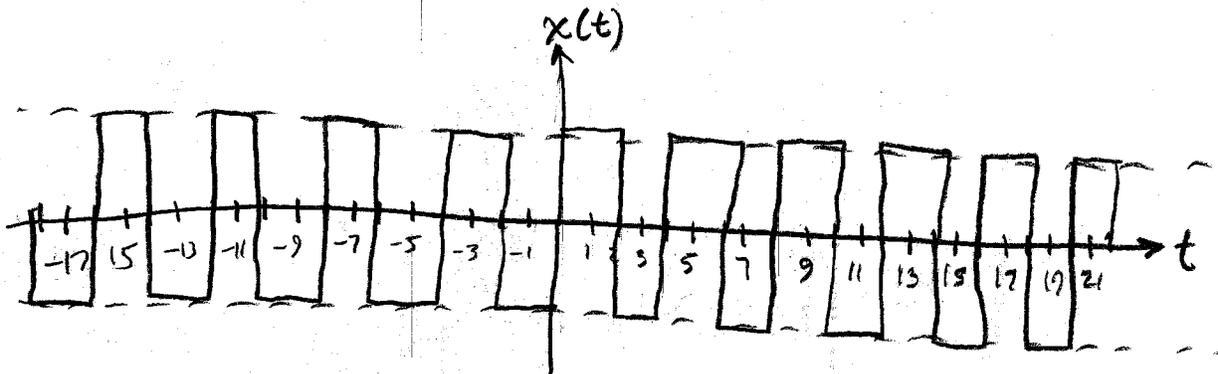
PERIODIC SIGNALS

→ Continuous time case:

DEF: if $\exists T \in \mathbb{R}, T > 0$, such that $x(t+T) = x(t)$
 $\forall t \in \mathbb{R}$, then the function $x(t)$ is called
periodic with period T .

Note: if $x(t)$ is periodic with period T , then $x(t)$ is also periodic with period $2T$ and with period NT for any integer $N > 0$.

EX: Periodic with period 4:



Also with period 8! And 16!

DEF: if T_0 is the smallest positive number such that $x(t+T_0) = x(t) \forall t \in \mathbb{R}$, then T_0 is called the Fundamental Period of $x(t)$.

→ Discrete time case:

DEF: if $\exists N \in \mathbb{N}$ s.t. $x[n+N] = x[n] \forall n \in \mathbb{Z}$,
then $x[n]$ is called periodic with period N .

Note: if $x[n]$ is periodic with period N , then $x[n]$ is also periodic with period kN , where k is any positive integer.

DEF: if N_0 is the smallest positive integer such that $x[n+N_0] = x[n] \forall n \in \mathbb{Z}$, then N_0 is called the Fundamental Period of $x[n]$.

NOTE: A signal that is not periodic is called "aperiodic".

NOTE: A continuous-time constant signal like $x(t) = 5$ is periodic with any positive real period. Since there is no unique smallest period, the fundamental period is undefined.

NOTE: A discrete-time constant signal like $x[n] = 5$ is periodic with any positive integer period.

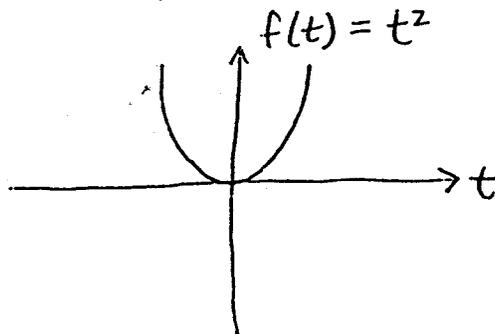
→ The fundamental period of a constant signal $x[n]$ is 1.

Even & Odd Signals

DEF: IF $f(t) = f(-t)$, THEN $f(t)$ IS CALLED AN EVEN FUNCTION.

EVEN FUNCTIONS ARE SYMMETRIC ABOUT THE VERTICAL AXIS:

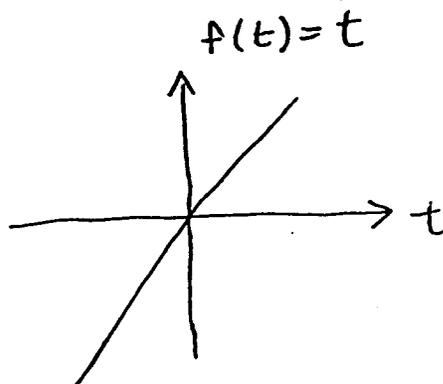
EX



DEF: IF $f(-t) = -f(t)$, THEN $f(t)$ IS CALLED AN ODD FUNCTION.

ODD FUNCTIONS ARE ANTISYMMETRIC ABOUT THE VERTICAL AXIS:

EX



- ANY FUNCTION $f(t)$ CAN BE WRITTEN AS THE SUM OF AN EVEN FUNCTION $Ev\{f(t)\}$ AND AN ODD FUNCTION $Od\{f(t)\}$.

$$Ev\{f(t)\} = \frac{1}{2} [f(t) + f(-t)]$$

$$Od\{f(t)\} = \frac{1}{2} [f(t) - f(-t)]$$

CHECK :

DOES $Ev\{f(t)\} + Od\{f(t)\} = f(t)$?

$$\begin{aligned} Ev\{f(t)\} + Od\{f(t)\} &= \frac{1}{2} [f(t) + f(-t)] + \frac{1}{2} [f(t) - f(-t)] \\ &= \frac{1}{2} [f(t) + f(-t) + f(t) - f(-t)] \\ &= \frac{1}{2} [2f(t)] = f(t) \checkmark \end{aligned}$$

IS $Ev\{f(t)\}$ AN EVEN FUNCTION?

$$\begin{aligned} Ev\{f(-t)\} &= \frac{1}{2} [f(\alpha) + f(-\alpha)] \Big|_{\alpha=-t} \\ &= \frac{1}{2} [f(-t) + f(t)] = \frac{1}{2} [f(t) + f(-t)] = Ev\{f(t)\} \checkmark \end{aligned}$$

IS $Od\{f(t)\}$ AN ODD FUNCTION?

$$\begin{aligned} Od\{f(-t)\} &= \frac{1}{2} [f(\alpha) - f(-\alpha)] \Big|_{\alpha=-t} \\ &= \frac{1}{2} [f(-t) - f(t)] = -\frac{1}{2} [f(t) - f(-t)] \\ &= -Od\{f(t)\} \checkmark \end{aligned}$$

- IF $x[-n] = x[n]$ FOR ALL VALUES OF n , THEN $x[n]$ IS AN EVEN SEQUENCE.

- IF $x[-n] = -x[n]$ FOR ALL VALUES OF n , THEN $x[n]$ IS AN ODD SEQUENCE.

- ANY SEQUENCE CAN BE WRITTEN AS THE SUM OF AN EVEN SEQUENCE $\{E_v\{x[n]\}\}$ AND AN ODD SEQUENCE $\{O_d\{x[n]\}\}$,

$$E_v\{x[n]\} = \frac{1}{2} [x[n] + x[-n]]$$

$$O_d\{x[n]\} = \frac{1}{2} [x[n] - x[-n]]$$

FACTS : $x[n] = E_v\{x[n]\} + O_d\{x[n]\}$.

$E_v\{x[n]\}$ IS EVEN.

$O_d\{x[n]\}$ IS ODD.

NOTE: IF $x(t)$ IS odd, THEN $x(0) = -x(0)$.

THIS MEANS $x(0) = 0$.

IF $x[n]$ IS odd, THEN $x[0] = 0$ TOO.

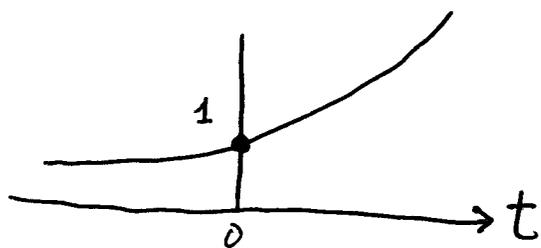
Exponential Signals: Continuous time

- We now consider signals of the form

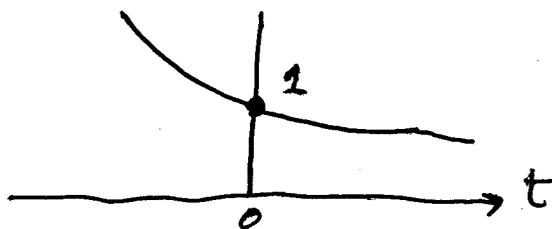
$$x(t) = e^{at}$$

- These are called "exponential signals".

- If $a \in \mathbb{R}$ and $a > 0$, $x(t)$ is a "growing exponential"; for $a < 0$, $x(t)$ is a "decaying exponential".



Growing



Decaying

- If a is purely imaginary, then $\exists \omega_0 \in \mathbb{R}$ such that $a = j\omega_0$.

→ In this case, $e^{at} = e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$.

$\text{Re}\{x(t)\} = \cos \omega_0 t$, a cosine with frequency ω_0 .

$\text{Im}\{x(t)\} = \sin \omega_0 t$, a sine with frequency ω_0 .

→ Since $\sin \theta = \cos(\theta - \frac{\pi}{2})$, the real and imaginary parts of $x(t) = e^{j\omega_0 t}$ are the same up to a phase shift of $\frac{\pi}{2}$ radians.

- Since the fundamental period of sine and cosine is 2π radians, the fundamental period of $\cos\omega_0 t$ and $\sin\omega_0 t$ is

$$T_0 = \left| \frac{2\pi}{\omega_0} \right|.$$

- Another way to look at the period:

T_0 is the smallest number such that

$$e^{j\omega_0 t} = e^{j\omega_0 (t + T_0)} = e^{j\omega_0 t} e^{j\omega_0 T_0}.$$

→ So T_0 is the smallest nonzero number such that $e^{j\omega_0 T_0} = 1$,

→ This means $e^{j\omega_0 T_0} = \cos\omega_0 T_0 + j\sin\omega_0 T_0 = 1$.

→ This means $\sin\omega_0 T_0 = 0$
 $\cos\omega_0 T_0 = 1$

⇒ So $\omega_0 T_0 = k2\pi$ for any $k \in \mathbb{Z}$.

→ The smallest nonzero period is when $k=1$. Then $\omega_0 T_0 = 2\pi$

$$\text{so } T_0 = \left| \frac{2\pi}{\omega_0} \right|.$$

- The signal $x(t) = e^{j\omega_0 t}$ is called a "complex exponential" or a "complex sinusoid".

→ conjugate symmetric

NOTE: For any complex number z , $z = a + jb$,

$$z + z^* = (a + jb) + (a - jb) = 2a$$

$$\rightarrow \text{So } \text{Re}[z] = \frac{z + z^*}{2}$$

$$z - z^* = (a + jb) - (a - jb) = 2jb$$

$$\rightarrow \text{So } \text{Im}[z] = \frac{z - z^*}{2j}$$

- Applying these formulas to $x(t) = e^{j\omega_0 t}$, we have

$$\cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

} another form of Euler's formula.

Note: It is standard to use " ω " for radian frequency and " f " for Hertzian frequency.

- Since $2\pi \text{ rad} = 1 \text{ Hz}$,

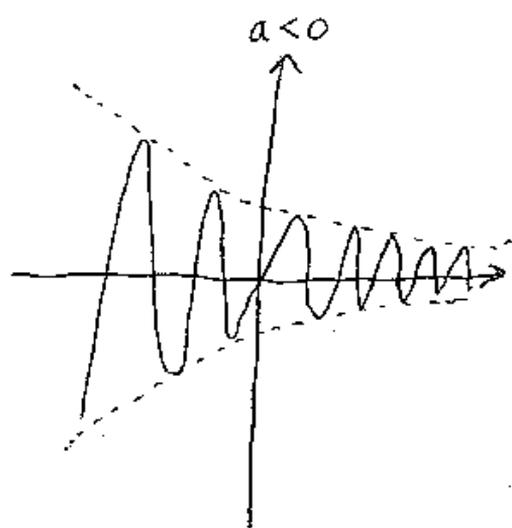
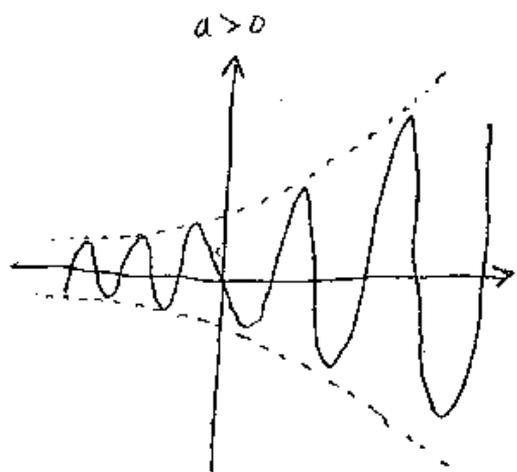
$$\omega = 2\pi f \quad \text{and} \quad f = \frac{\omega}{2\pi}$$

1.44

PAGE ~~1.46~~

- IF YOU MULTIPLY A COMPLEX EXPONENTIAL BY A REAL EXPONENTIAL, YOU GET SOMETHING CALLED A "DAMPED EXPONENTIAL" OR A "DAMPED SINUSOID":

$$e^{at} e^{j\omega_0 t} = e^{at} [\cos \omega_0 t + j \sin \omega_0 t]$$



- In the most general case, if $C \in \mathbb{C}$ and $a \in \mathbb{C}$, where $C = |C| e^{j\theta}$ and $a = \sigma + j\omega_0$ ($\theta, \sigma, \omega_0 \in \mathbb{R}$), we have

$$\begin{aligned}
 C e^{at} &= |C| e^{\sigma t} \left\{ \cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta) \right\} \\
 &= |C| e^{\sigma t} e^{j(\theta + \omega_0 t)}
 \end{aligned}$$

Exponential Signals: Discrete Time

- A discrete time exponential signal takes the form

$$x[n] = Ce^{\beta n},$$

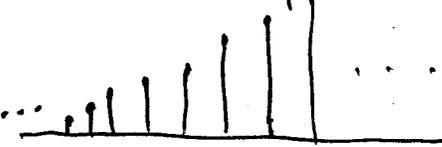
where C and β may be real, imaginary, or complex.

- This is usually written as

$$x[n] = C\alpha^n, \text{ where } \alpha = e^{\beta}.$$

- When α and C are real, there are 4 possible behaviors:

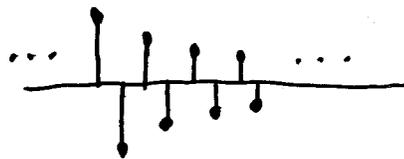
1. $\alpha > 1$: growing exponential



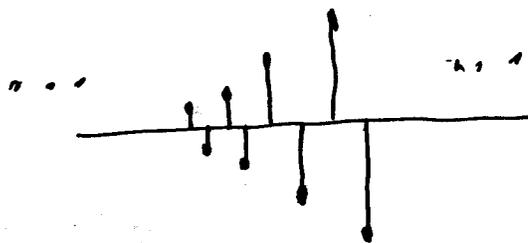
2. $0 < \alpha < 1$: decaying exponential



3. $-1 < \alpha < 0$: alternating decaying exponential



4. $\alpha < -1$: alternating growing exponential



- If $C=1$ and β is pure imaginary, the signal

$$x[n] = Ce^{\beta n} = e^{j\omega_0 n} = \cos \omega_0 n + j \sin \omega_0 n$$

is a discrete time complex exponential. \rightarrow *conjugate symmetric*

\rightarrow You can think of $x[n]$ as containing samples of the continuous time signal $x(t) = e^{j\omega_0 t}$.

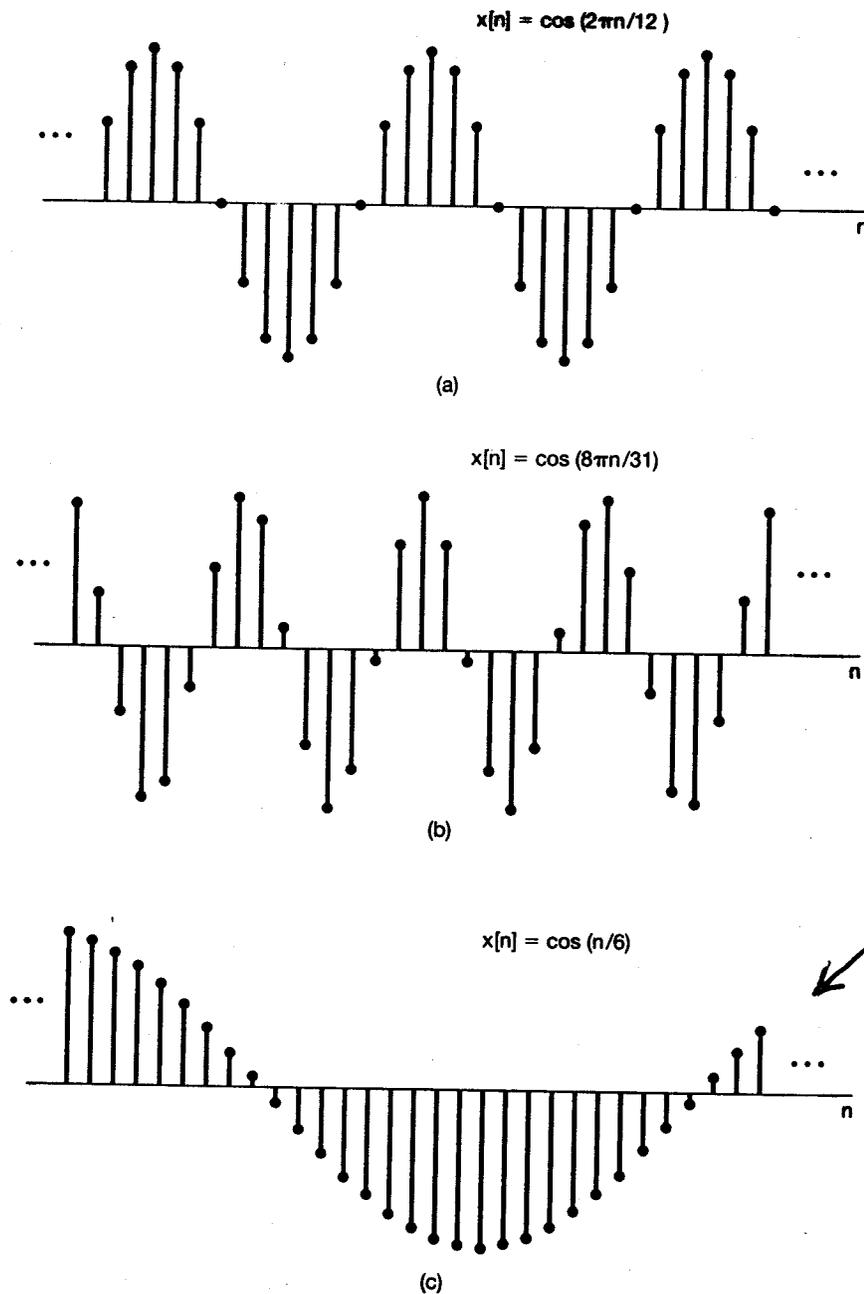
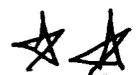


Figure 1.25 Discrete-time sinusoidal signals.



This signal is not periodic.

The samples don't line up with the period boundaries of the continuous signal.

- $\cos \omega_0 n$, $\sin \omega_0 n$, and $e^{j\omega_0 n}$ are examples of discrete time signals that have infinite energy but finite power.

- When C and α are complex, the signal $x[n] = C\alpha^n$ generally has real and imaginary parts that are discrete time damped sinusoids.

→ if $C = |C|e^{j\theta}$ and $\alpha = |\alpha|e^{j\omega_0}$,
where $\theta, \omega_0 \in \mathbb{R}$, then

$$x[n] = C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + |C||\alpha|^n j \sin(\omega_0 n + \theta).$$

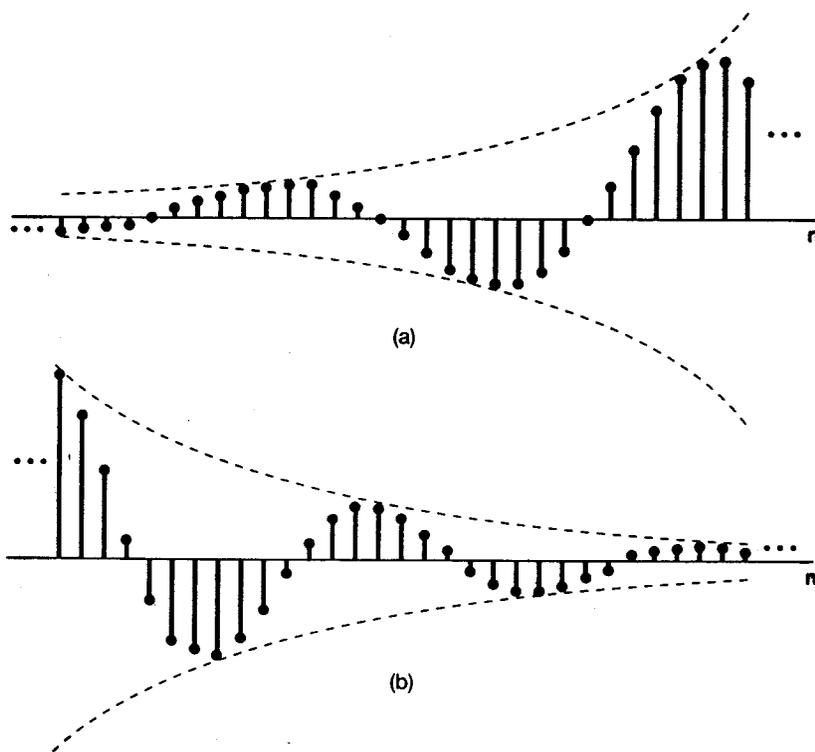
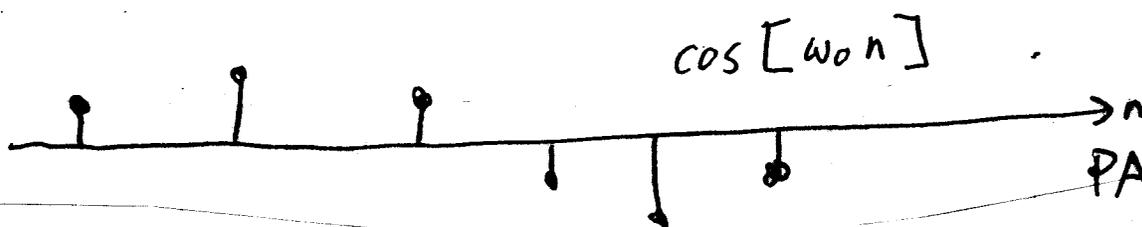
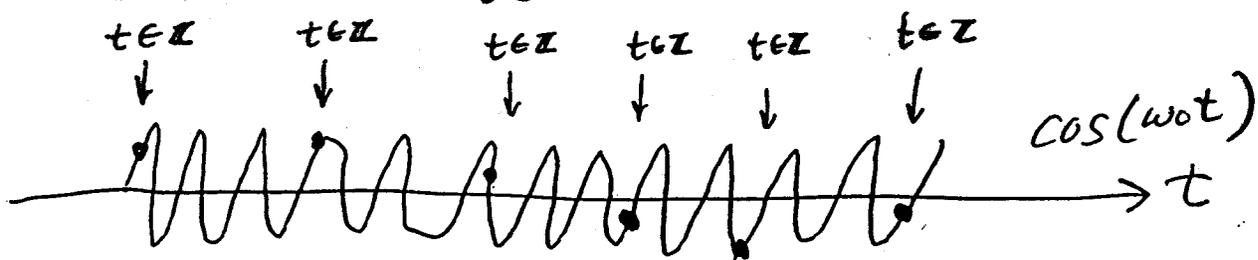


Figure 1.26 (a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

IMPORTANT DIFFERENCES BETWEEN DISCRETE AND CONTINUOUS SINUSOIDS

- To understand the differences, consider
 - a continuous time signal $x(t) = \cos \omega_0 t$
 - a discrete time signal $x[n] = \cos \omega_0 n$, obtained by sampling $x(t)$.
- As ω_0 gets bigger and bigger, $x(t)$ oscillates faster and faster.
- $x[n]$ also oscillates faster up to a point, but then, as ω_0 gets bigger, $x[n]$ oscillates slower!!
- Why?
 - If ω_0 is big, $\cos \omega_0 t$ may go through several cycles between places where t is an integer.
 - Then the graph of $\cos \omega_0 n$ doesn't look much like the graph of $\cos \omega_0 t$:

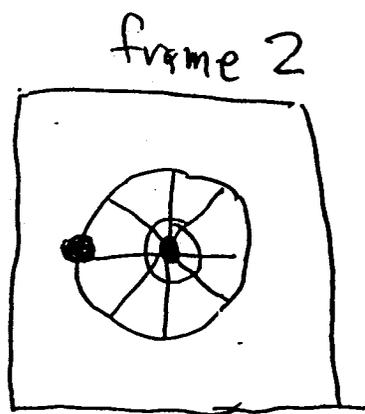
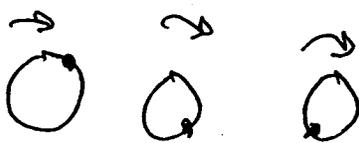
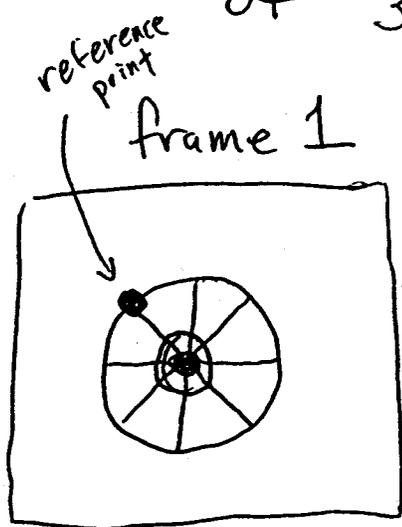


- Notice how the high frequency continuous-time sinusoid turned into a slowly varying discrete-time waveform.

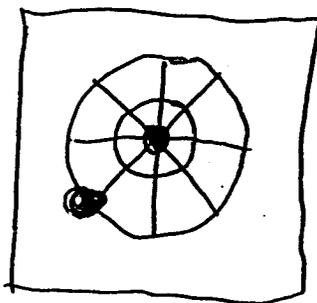
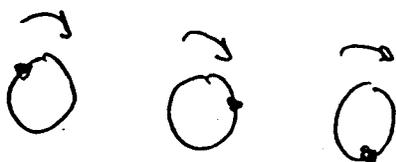
- This phenomenon is called "ALIASING".

- It is the same effect that makes wagon wheels look like they are spinning slowly backwards on film.

→ Standard video has a frame rate of 30 frames per second.



frame 3



Because the wheel almost gets all the way around between each frame, it appears to be turning slowly backwards,

- All of the unique discrete cosines can be generated with $0 \leq \omega_0 \leq \pi$.

→ if $-\pi \leq \omega_0 \leq 0$, then $\cos \omega_0 n = \cos(-\omega_0 n)$,
and $0 \leq -\omega_0 \leq \pi$.

→ if $\pi \leq \omega_0 \leq 2\pi$, then $\cos \omega_0 n = \cos[(2\pi - \omega_0)n]$,
and $0 \leq 2\pi - \omega_0 \leq \pi$.

→ if $-2\pi \leq \omega_0 \leq -\pi$, then

$$\cos \omega_0 n = \cos(-\omega_0 n) = \cos[(\omega_0 + 2\pi)n],$$

and $0 \leq \omega_0 + 2\pi \leq \pi$.

→ Thus, when ω_0 is outside $[0, \pi]$, the signal $\cos \omega_0 n$ has the exact same values as another signal with frequency inside $[0, \pi]$.

→ In fact, $\cos \omega_0 n = \cos[(\omega_0 + 2\pi k)n]$ for any $k \in \mathbb{Z}$.

- Since $\sin \omega_0 n = -\sin(-\omega_0 n)$, we can generate all the unique discrete sines with frequencies

$$-\pi \leq \omega_0 \leq \pi.$$

→ For any $k \in \mathbb{Z}$, $\sin[(\omega_0 + 2\pi k)n] = \sin \omega_0 n$.

- For these reasons, we only need to consider discrete frequencies ω_0 in the range $-\pi \leq \omega_0 \leq \pi$.

→ For any ω_0 outside this range, $\sin \omega_0 n$ is the same signal as another sine with frequency inside $[-\pi, \pi]$.

→ For any ω_0 outside this range, $\cos \omega_0 n$ is the same signal as another cosine with frequency inside $[0, \pi]$.

- People sometimes say that "discrete frequencies greater than π do not exist!"

→ This isn't exactly true. The frequencies do exist, but they give signals with values that are identical to those that would be obtained with a lower frequency.

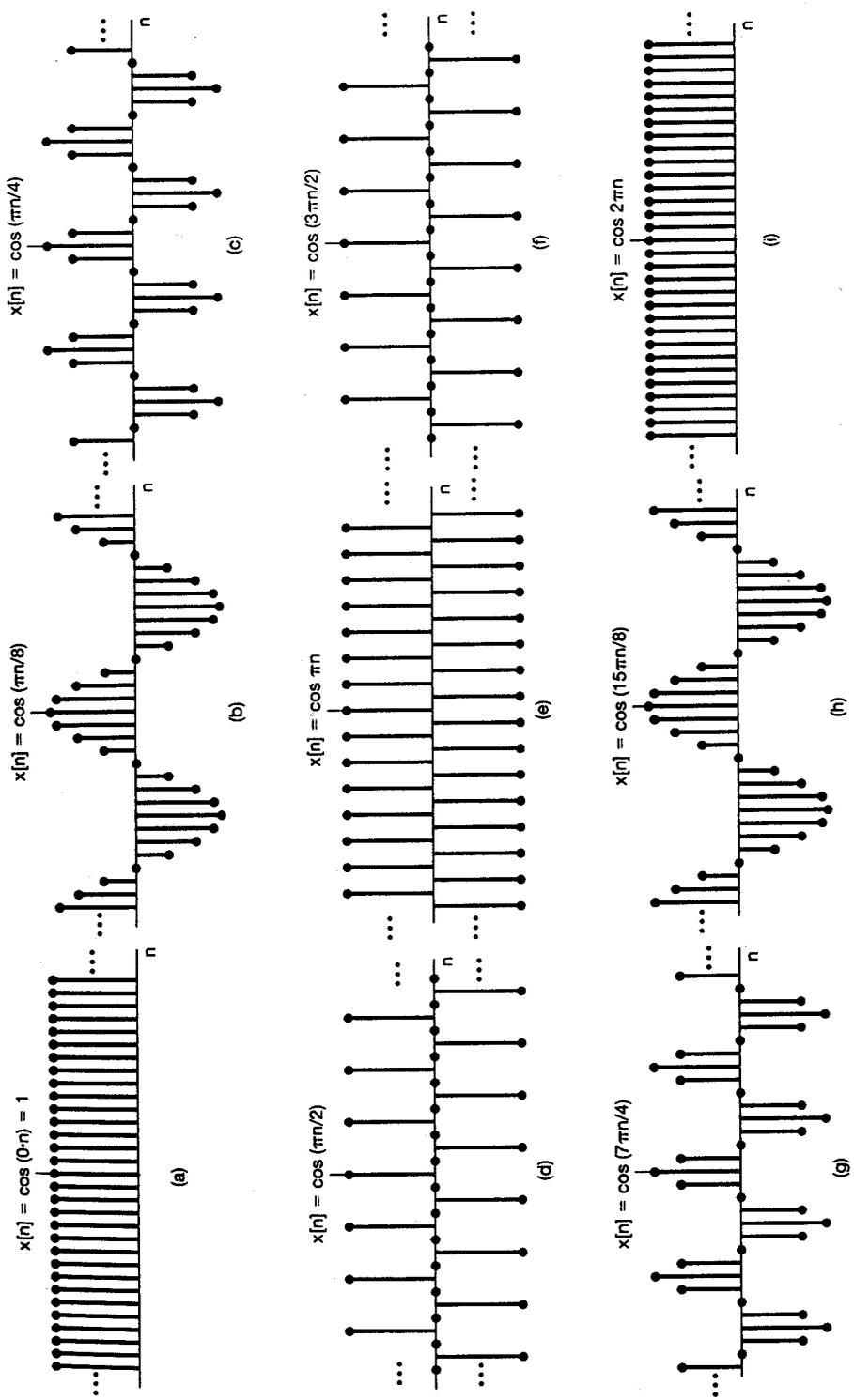


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

1.53
PAGE 157

- Another difference: the discrete time signals

$\cos \omega_0 n$, $\sin \omega_0 n$, and $e^{j\omega_0 n}$ are
periodic only if $\frac{\omega_0}{2\pi} \in \mathbb{Q}$.

→ otherwise, the samples $\cos \omega_0 n$ fall at different places in the continuous signal $\cos \omega_0 t$ in each period of $\cos \omega_0 t$.

→ In other words, the integer points are at different points in the period of $\cos \omega_0 t$ in each period, and no two "periods" of $\cos \omega_0 n$ are exactly the same.

→ The samples don't "line up" from period to period.

- If $\frac{\omega_0}{2\pi}$ is rational, then $\cos \omega_0 n$, $\sin \omega_0 n$, and $e^{j\omega_0 n}$ are periodic.

→ In this case, the samples fall at the same points in each ^{super}period of $\cos \omega_0 t$. ← (It may take several periods of $\cos \omega_0 t$ to

→ For $\frac{\omega_0}{2\pi}$ to be rational, ω_0 must contain ^{make one period of} the factor π . $\cos \omega_0 n$)

1.54

PAGE ~~1.59~~

- If $\omega_0/2\pi \in \mathbb{Q}$, you can find the fundamental period by writing $\frac{\omega_0}{2\pi} = \frac{m}{N}$, where $m, N \in \mathbb{Z}$.

→ if $\frac{m}{N}$ is in reduced form, then m and N have no common factors and N is the fundamental period.

EX: $x[n] = \cos\left(\frac{8\pi}{31}n\right)$.

$$\omega_0 = \frac{8\pi}{31} \Rightarrow \frac{\omega_0}{2\pi} = \frac{8}{62} = \frac{4}{31} = \frac{m}{N}$$

Fundamental period = $N = 31$.

{ each $m=4$ periods of $\cos\omega_0 t$ make one period of $\cos\omega_0 n$ }

EX: $x[n] = \cos\frac{n}{6}$.

$$\omega_0 = \frac{1}{6} \Rightarrow \frac{\omega_0}{2\pi} = \frac{1}{12\pi} \notin \mathbb{Q} \Rightarrow \underline{\text{NOT PERIODIC.}}$$

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

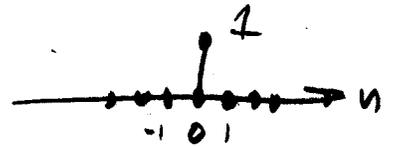
$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m .
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined 2 $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

* Assumes that m and N do not have any factors in common.

Discrete-Time Unit Impulse and Unit Step

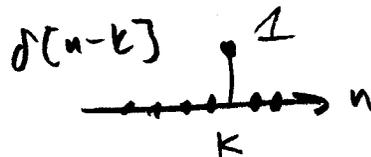
- we have already seen the "Kronecker delta" or "discrete time unit impulse":

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{other} \end{cases}$$

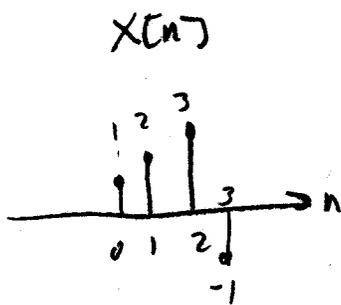


"turned on at zero"

- The translate $\delta[n-k]$ is "turned on" at $n=k$:



- Clearly, any discrete-time signal $x[n]$ can be written as a linear combination (weighted sum) of the set (basis) $\{\delta[n-k]\}_{k \in \mathbb{Z}}$.



$$= \frac{1}{0} + \frac{2}{1} + \frac{3}{2} + \frac{-1}{-1}$$

$$= \delta[n] + 2\delta[n-1] + 3\delta[n-2] - \delta[n-3]$$

- You can do this by taking dot products:

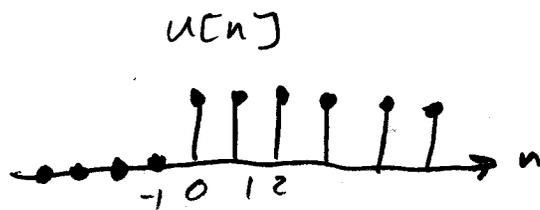
$$c_k = \langle x[n], \delta[n-k] \rangle = \sum_{n=-\infty}^{\infty} x[n] \delta[n-k] = x[k] \quad \text{PAGE 1.56}$$

- When you work on the ^{time} graph of a discrete-time signal, you are representing the signal in terms of the basis $\{\delta[n-k]\}_{k \in \mathbb{Z}}$

- This is called discrete-time time domain analysis.

- Discrete-time unit step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



NOTE: $u[n] = \sum_{k=-\infty}^n \delta[k]$

Also, from graph using dot products: $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

Continuous-Time Unit Step and Unit Impulse

- This is the part of the story where we run into some trouble modeling signals with functions.

- The mathematically correct way to model these signals is using a related type of math objects called distributions.

⇒ a distribution defines a signal by how it behaves under integrals and in dot products.

$$\text{Recall: } \langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

- For example, the classical definition of the Dirac delta ("delta function") is:

$$\int_{-\infty}^{\infty} x(t) \delta(t) dt = x(0)$$

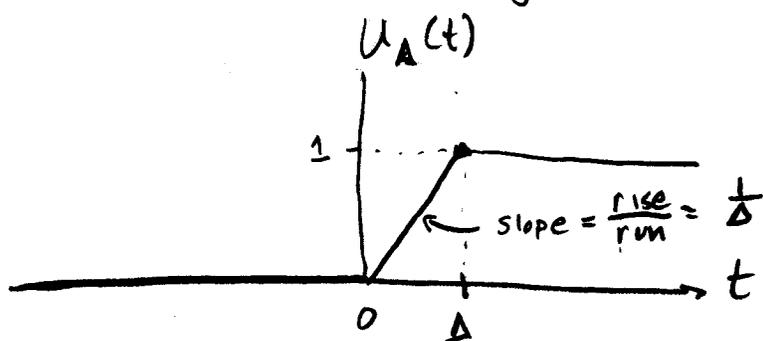
- Note that there exists no function with this property. [$\delta(t) = 0$ a.e.]

→ more on $\delta(t)$ shortly.

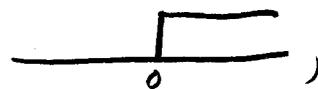
- The distribution concept frees us from having to define a function at every t (say what t 's buddy is).

- We don't have time to cover distribution theory rigorously, so we will be informal.

- Consider a whole class of signals $u_{\Delta}(t)$ with Δ small:



- On the scope, they all look the same

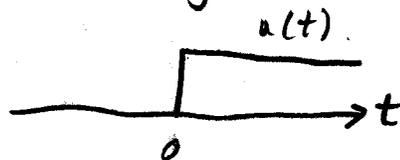


and the exact values of the signal around $t=0$ aren't important.

- For the time scale we are interested in, we can use one model for this whole class of signals.

→ It is

$$u(t) = \lim_{\Delta \rightarrow 0} u_{\Delta}(t)$$



- The book defines it as

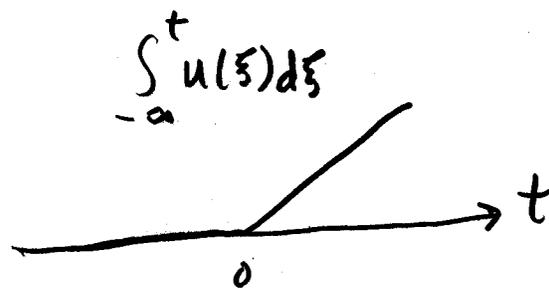
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

⇒ NOTE: This is not a function because it doesn't tell you what the value of $u(0)$ is.

→ But under an integral or in a dot product, the value of $u(t)$ at one point doesn't matter.

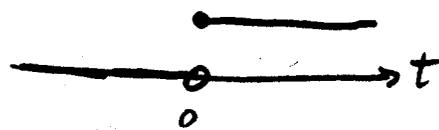
→ what it is is called a distribution or generalized function.

- The integral of $u(t)$ is:

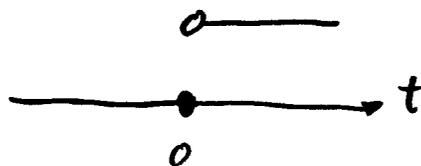


- Other definitions people use for $u(t)$ include:

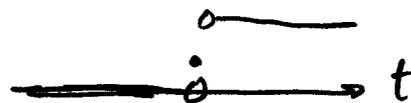
$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$$



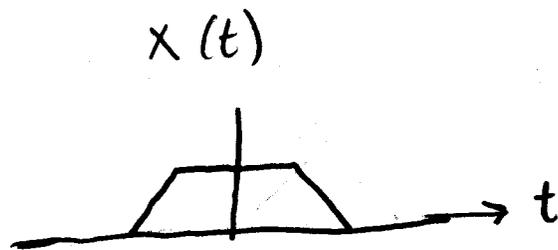
$$u(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$



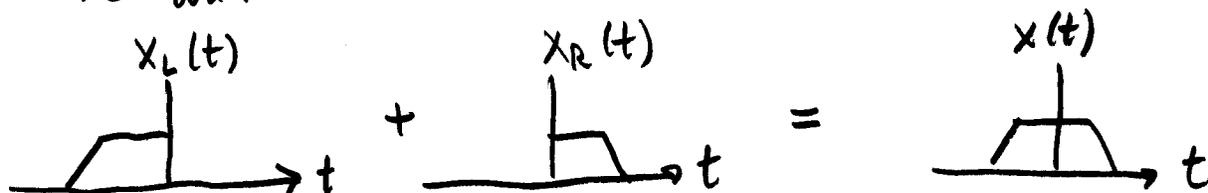
- None of these definitions will work for all cases we will see in this class.

\Rightarrow when you need the value of $u(t)$ at $t=0$, you just have to use your head.

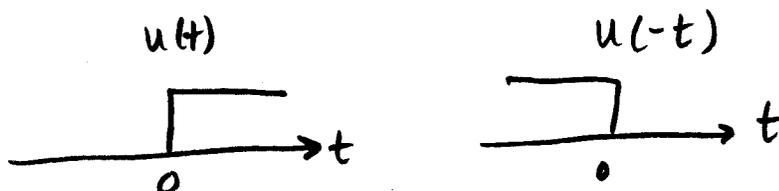
EX:



- we want to break up $x(t)$ into $x_L(t)$ and $x_R(t)$:



NOTE:



- So we should be able to break the signal up using

$$x_L(t) = u(-t)x(t)$$

$$x_R(t) = u(t)x(t)$$

- But does this work? Does

$$x(t) \stackrel{?}{=} u(-t)x(t) + u(t)x(t) \quad ??$$

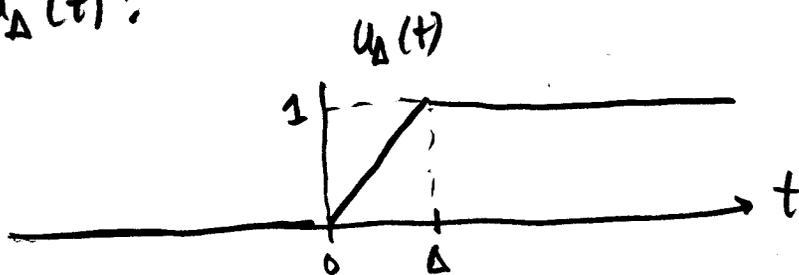
- It depends how you define $u(0)$.

- For the above to work, we have to let $u(0) = 1$ for one of the unit steps and $u(0) = 0$ for the other... or let $u(0) = \frac{1}{2}$ for both.

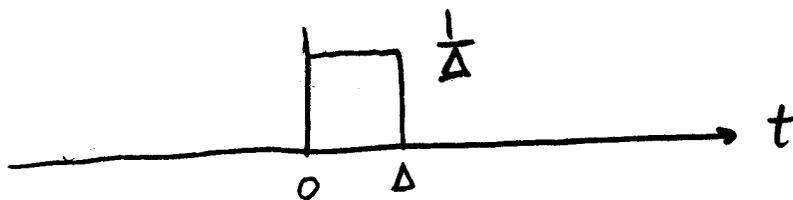
\Rightarrow No general rule will always work. You just have to use your head.

- Now back to $u_{\Delta}(t)$... let's consider its derivative

$$d_{\Delta}(t) = \frac{d}{dt} u_{\Delta}(t):$$



$$d_{\Delta}(t) = \frac{d}{dt} u_{\Delta}(t)$$

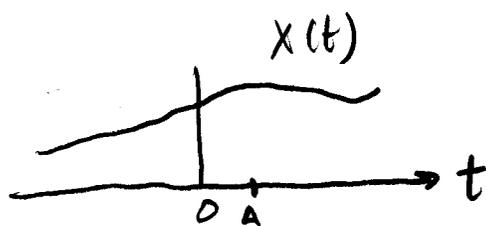


- For Δ small, all the signals $d_{\Delta}(t)$ in the class are short unit-area pulses.

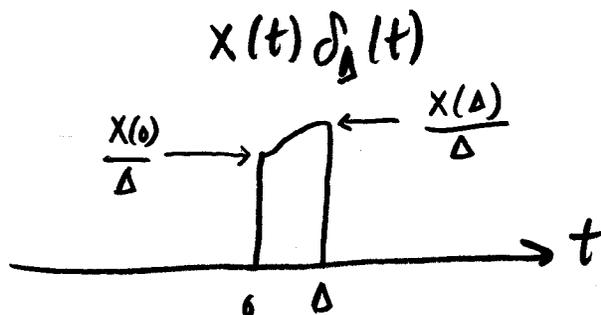
- On the scope, they all look like this: $\frac{1}{0}$

- But have unit area: $\int_{-\infty}^{\infty} d_{\Delta}(t) dt = 1$.

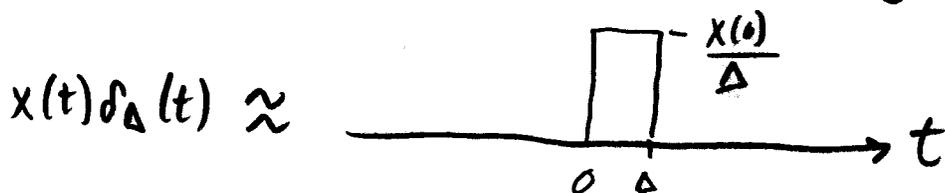
- Still with Δ small, look at how $d_{\Delta}(t)$ behaves in a dot product...



$$\langle x(t), \delta_{\Delta}(t) \rangle = \int_{-\infty}^{\infty} x(t) \delta_{\Delta}(t) dt$$



- If $x(t)$ is reasonably smooth and Δ is small, then $x(\Delta) \approx x(0)$, and there is little error in approximating the product by



- The dot product is the area under this, and we have

$$\langle x(t), \delta_{\Delta}(t) \rangle = \int_{-\infty}^{\infty} x(t) \delta_{\Delta}(t) dt \approx \frac{x(0)}{\Delta} \cdot \Delta = x(0).$$

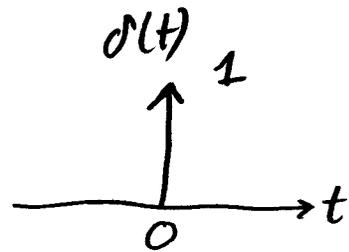
- So putting it all together and letting $\Delta \rightarrow 0$, we come to the classical definition of the Dirac Delta:

$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t)$$

with these properties:

$$\left. \begin{aligned} \delta(t) &= 0, \quad t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t) dt &= 1 \end{aligned} \right\} \text{implies infinite height}$$

$$\int_{-\infty}^{\infty} x(t) \delta(t) dt = x(0)$$



\Rightarrow of course this is nonsense as a function, because there is no area under a point.

\rightarrow Always remember that $\delta(t)$ is a generalized function that we use to model any short, tall, unit area pulse like signal.

- Applying time shifting, we get the so called "sifting property" of $\delta(t)$:

$$\int_{-\infty}^{\infty} \delta(t-t_0) x(t) dt = x(t_0)$$

- NOTE: $\int_{-\infty}^{\infty} \delta(t-z) x(t) dt = x(z)$

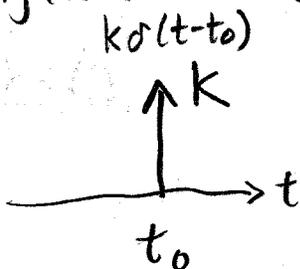
but $\int_{-\infty}^0 \delta(t-z) x(t) dt = 0$ (must cross the place where the delta is "turned on").

- Be careful when you have the product of a signal with a Dirac delta:

$$x(t) \delta(t-t_0) = x(t_0) \delta(t-t_0)$$

↑ a signal
↑ a distribution
↑ a number.
↑ a distribution

- Weighted Dirac delta:



$$\int_{-\infty}^{\infty} k \delta(t-t_0) dt = k \int_{-\infty}^{\infty} \delta(t-t_0) dt = k \cdot 1 = k$$

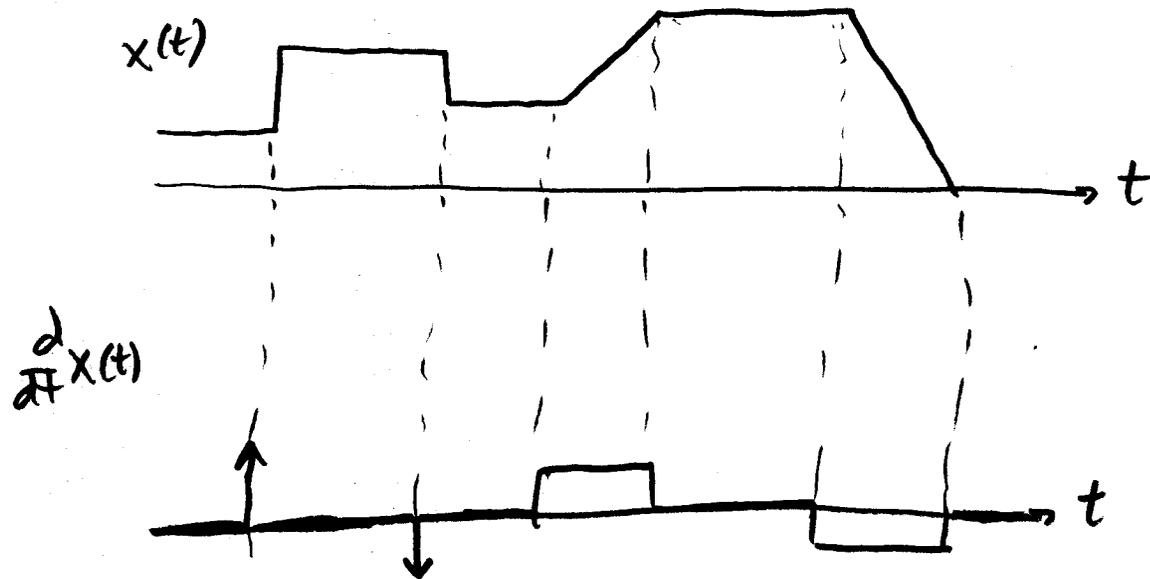
- We have that $\frac{d}{dt} u_{\Delta}(t) = \delta_{\Delta}(t)$,

$$\lim_{\Delta \rightarrow 0} u_{\Delta}(t) = u(t) ; \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t) = \delta(t)$$

- Putting it all together

$$\lim_{\Delta \rightarrow 0} \left(\frac{d}{dt} u_{\Delta}(t) = \delta_{\Delta}(t) \right) \Rightarrow \frac{d}{dt} u(t) = \delta(t)$$

- This allows us, in a distributional sense, to represent the derivatives of signals that are not, strictly speaking, differentiable:



- Time scaling for Dirac delta:

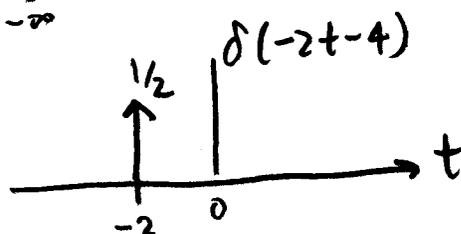
$$\delta(at) = \frac{1}{|a|} \delta(t)$$

\Rightarrow So,

$$\int_{-\infty}^{\infty} x(t) \delta(-2t-4) dt = \int_{-\infty}^{\infty} x(t) \delta[-2(t+2)] dt$$

$$= \int_{-\infty}^{\infty} x(t) \frac{1}{2} \delta(t+2) dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x(t) \delta(t+2) dt = \frac{1}{2} x(-2)$$



- The derivative of $\delta(t)$ is called the "unit doublet".

$$\delta'(t) = \frac{d}{dt} \delta(t)$$

- It's main property is

$$\langle x(t), \delta'(t) \rangle = \int_{-\infty}^{\infty} x(t) \delta'(t) dt = -x'(0)$$

- You can derive this by differentiating $\delta_{\Delta}(t)$ and taking the limit as $\Delta \rightarrow 0$.

- Higher derivatives of $\delta(t)$ are defined by the property

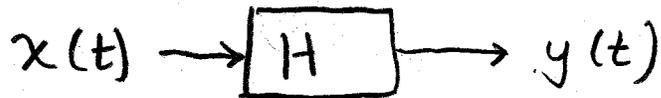
$$\langle x(t), \delta^{(k)}(t) \rangle = \int_{-\infty}^{\infty} x(t) \delta^{(k)}(t) dt = (-1)^k x^{(k)}(0).$$

- Now, back to the book!

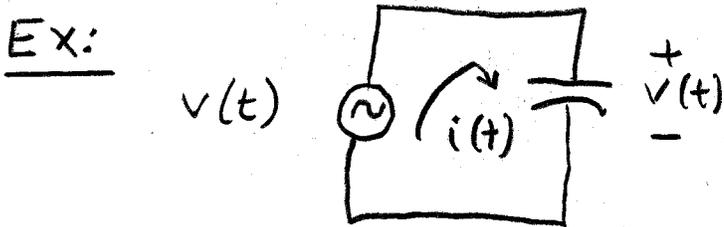
- Recall: a system is something that inputs a signal and outputs a signal.

→ It is a mapping (a function) from one set of signals to another set of signals.

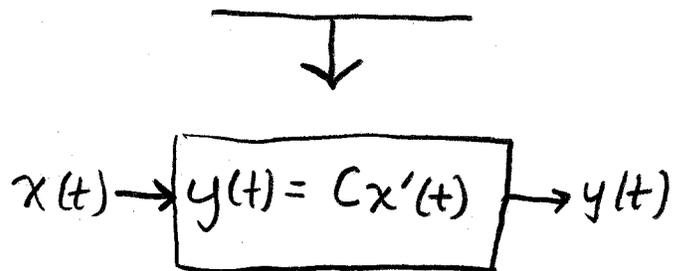
- A continuous-time system:



Ex: $y(t) = \int_{t-5}^t x(\tau) d\tau$



input: $x(t) = v(t)$
output: $y(t) = i(t)$



- A discrete-time system:

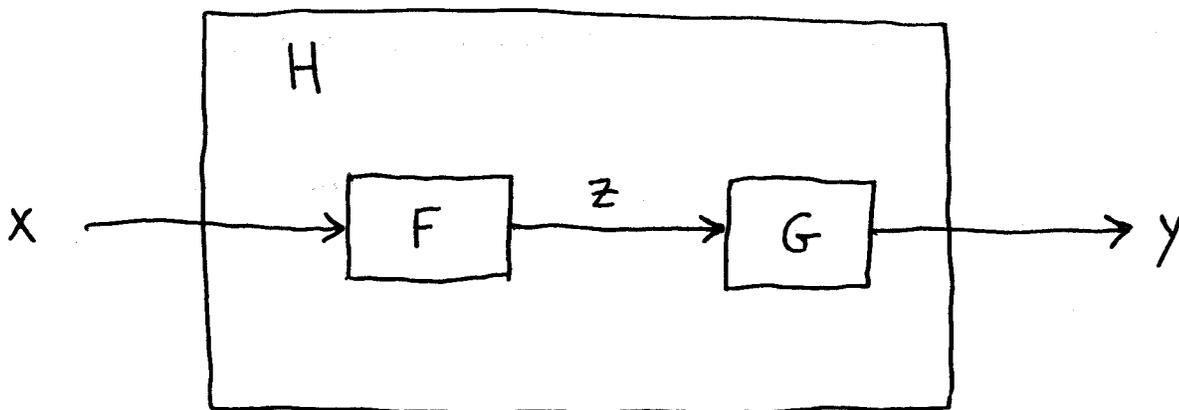


EX: $y[n] = \sum_{k=n-5}^n x[k]$

Interconnections of Systems

- Back on page 1.83, we saw that systems often contain parts that can themselves be considered as systems.
- We now consider some common ways that small systems, or "subsystems" are interconnected to form bigger systems.
- We will just write "x", "y", etc., for the signals.
 - For a continuous-time system, they would be $x(t)$ and $y(t)$,
 - For a discrete-time system, they would be $x[n]$ and $y[n]$.

"Cascade" or "Series" Connection:

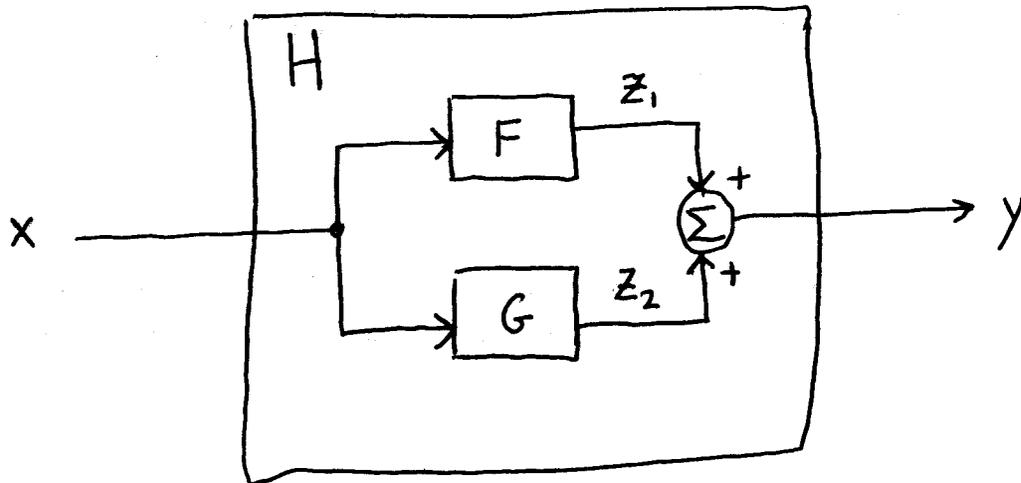


- System F inputs x and outputs z .
- System G inputs z and outputs y .
- System H inputs x and outputs y .
- H is a series connection of F and G.

1.70

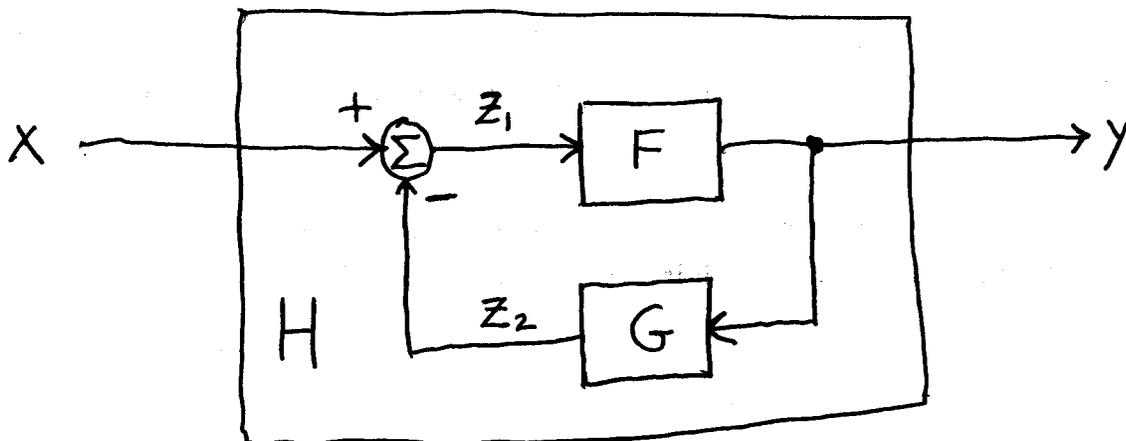
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Parallel Connection :



- System F inputs x and outputs z_1 .
- System G inputs x and outputs z_2 .
- System H inputs x and outputs $y = z_1 + z_2$.
- H is a parallel connection of F and G.

Feedback Connection :



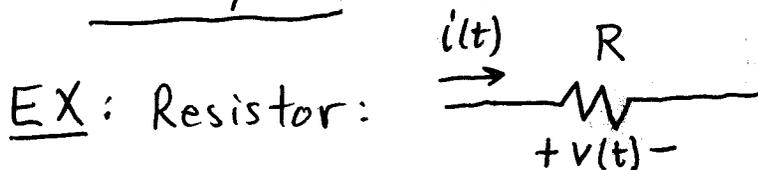
- System H inputs x and outputs y .
- System G inputs y and outputs z_2 .
- System F inputs $z_1 = x - z_2$ and outputs y .
- H is a feedback connection of F and G.

System Properties

- We will now examine several properties of systems.
- To show that a system has one of the properties, it is generally necessary to show that the property holds for all possible input signals.
 - To do this, you usually start with an arbitrary signal $x(t)$ or $x[n]$, and see if the property holds.
- To show that a system does not have one of the properties, it is sufficient to construct just one input signal for which the property fails.

MEMORY

- If the output at any time depends only on the input at that same time, and not on the value of the input at other times, then the system is memoryless.



input: $x(t) = i(t)$, current
output: $y(t) = v(t)$, voltage

$$y(t) = Rx(t)$$

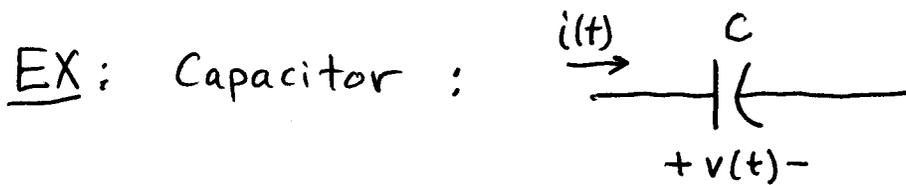
→ output at time t depends only on input at time t , and not at other times.

EX: $y[n] = 5x[n]$

- Output at time n depends only on the input at time n , and not on the value of the input at other times.

- A system that is not memoryless is said to have memory.

→ For a system with memory, the output at any one time can depend on the values of the input at that time and at other times too.



input: $x(t) = i(t)$, current
output: $y(t) = v(t)$, voltage

$$i(t) = C \frac{d}{dt} v(t)$$

→ Apply fundamental theorem of calculus:

$$v(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau$$

$$y(t) = \frac{1}{C} \int_{-\infty}^t x(\tau) d\tau$$

→ Output at time t depends on present value of input and all past values of input.

- This system has memory.

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EX: $y[n] = \sum_{k=n-5}^n x[k]$

→ output $y[n]$ depends on inputs $x[n]$, $x[n-1]$, $x[n-2]$, $x[n-3]$, $x[n-4]$, and $x[n-5]$.

→ This system has memory.

EX: $y[n] = x[n+30]$.

- In this case, the output at time n depends on a future value of the input.

- If this system could be built, we would all get rich on the stock market!

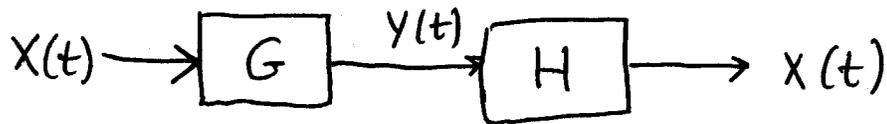
Invertibility

- Recall that a system is an operator: it's a function that maps input signals to output signals.

- If the mapping is one-to-one, then the system is invertible.

→ This means that no two distinct input signals produce the same output signal.

- If a system G is invertible, then it is possible to construct an inverse system H such that
 - If $x(t)$ is input to G to produce output $y(t)$
 - Then $y(t)$ is input to H , the output of H is the original signal $x(t)$:



- This is used often in communications.

- Suppose H represents the distortion that occurs when a communications signal passes through a communications channel.
- A common practice is to construct an inverse system G .
- The communications signal $x(t)$ is processed with G before being sent through the channel.
- The processed signal $y(t)$ is transmitted down the channel.
- The received signal is the same as the original signal $x(t)$.
- In this case, the system G is called a channel equalizer.

- General procedure to check if system is invertible:

① Try to solve the I/O relation for the input signal.

→ If you can, then switch the names of the input and output signals. That's the I/O relation for the inverse system.

→ If you can't, then use the "reason" to help you construct two input signals that both make the same output signal.

EX: $y(t) = 5x(t) \mapsto x(t) = \frac{1}{5}y(t)$

- The I/O relation for the inverse system is: $y(t) = \frac{1}{5}x(t)$.

EX: $y[n] = \sum_{k=-\infty}^n x[k] = \sum_{k=-\infty}^{n-1} x[k] + x[n] = y[n-1] + x[n]$.

so $y[n] = y[n-1] + x[n] \mapsto x[n] = y[n] - y[n-1]$

- The I/O relation for the inverse system is: $y[n] = x[n] - x[n-1]$

EX: $y[n] = 0$. Note: this means that the system maps every input signal to the output signal $y[n] = 0$

→ given the output signal, it is impossible to tell which input signal produced it,

→ we can't solve the I/O relation for the input signal.

→ The system is not invertible.

EX: $y(t) = x^2(t)$

→ This system does not have an inverse. Given $y(t)$, we can't tell if the input was $x(t) = \sqrt{y(t)}$ or $x(t) = -\sqrt{y(t)}$. They both make the same output.

CAUSALITY:

- If the value of the output signal at any given time does not depend on future values of the input signal, then the system is causal.
- This means that the value of the output signal at time t or n depends only on the past and present values of the input signal... not on future values of the input signal.

EX: averagers:

$$y(t) = \frac{1}{5} \int_{t-5}^t x(\tau) d\tau$$

$$y[n] = \frac{1}{5} \sum_{k=n-4}^n x[k]$$

→ Both causal because the present value of the output signal depends only on past and present values of the input signal... not on future values of the input signal.

EX: centered averagers:

$$y(t) = \frac{1}{5} \int_{t-2.5}^{t+2.5} x(\tau) d\tau$$

$$y[n] = \frac{1}{5} \sum_{k=n-2}^{n+2} x[k]$$

→ Both non-causal. Because $y(0)$ depends on the future input value $x(2)$ in the first case and $y[2]$ depends on the future input value $x[3]$ in the second case.

EX: $y(t) = \int_t^{\infty} x(\tau) d\tau$

→ not causal, obviously.

→ In this case, the present value of the output depends only on the present and future values of the input signal, but not on the past.

→ Such a system is called anticausal.

- What good are non-causal and anticausal systems?

- They can be used when the entire input signal is available at once.

EX:
- sonar data stored on disk.
- Audio data stored on CD.

→ you can read in the whole input signal and use non-causal processing.

- Also, consider an image $x[m, n]$.

- The whole picture is acquired at once.

- There is no reason that the processing should be causal with respect to the spatial coordinates m and n .

STABILITY:

DEF: A signal $x(t)$ is bounded if $\exists B \in \mathbb{R}$,
 $0 \leq B < \infty$, such that

$$|x(t)| \leq B \quad \forall (t).$$

DEF: A signal $x[n]$ is bounded if $\exists B \in \mathbb{R}$,
 $0 \leq B < \infty$, such that

$$|x[n]| \leq B \quad \forall n.$$

DEF: A system is stable if every bounded input produces a bounded output.

Note: This kind of stability is called "Bounded Input Bounded Output" stability, or "BIBO" stability.

- There are other kinds of stability, but we will not use them in this class.

→ For example, there is another kind that says the system is stable only if it is BIBO stable and, in addition, all internal signals inside the system are also bounded.

EX: $y[n] = \sum_{k=n-5}^n x[k].$

CLAIM: This system is (BIBO) stable.

PROOF: Let $x[n]$ be a bounded input signal.

- Then $\exists B \in \mathbb{R}, 0 \leq B < \infty$, s.t. $|x[n]| \leq B \forall n.$

- Furthermore,

$$\begin{aligned} |y[n]| &= \left| \sum_{k=n-5}^n x[k] \right| \leq \sum_{k=n-5}^n |x[k]| \quad \left(\begin{array}{l} \text{Triangle} \\ \text{inequality} \\ \text{for sums} \end{array} \right) \\ &\leq \sum_{k=n-5}^n B \\ &= 6B \end{aligned}$$

- Therefore, $|y[n]| \leq 6B$ and the system is stable. QED

(if the input signal is bounded by B , then the output signal is bounded by $6B$. So every bounded input signal produces a bounded output signal.)

EX: $y(t) = \int_{-\infty}^t x(\tau) d\tau$

- This system is not stable.

Proof: - Let $x(t) = u(-t)$.

- Then $x(t)$ is bounded, because
 $|x(t)| \leq 1 \quad \forall t \in \mathbb{R}$

- When $t=0$,

$$y(0) = \int_{-\infty}^0 u(-t) dt = \int_{-\infty}^0 1 dt \rightarrow \infty,$$

So $y(t)$ is not bounded.

QED.

EX: $y(t) = \int_{t-2}^{t+2} x(\tau) d\tau$

- Suppose $x(t)$ is a bounded input.

- Then $\exists B \in \mathbb{R}, 0 \leq B < \infty$, such that $|x(t)| \leq B \quad \forall t$.

- Then

$$|y(t)| = \left| \int_{t-2}^{t+2} x(\tau) d\tau \right| \leq \int_{t-2}^{t+2} |x(\tau)| d\tau$$

\uparrow Triangle Inequality for integrals

$$\leq \int_{t-2}^{t+2} B d\tau = 4B.$$

- Therefore, $|y(t)| \leq 4B \quad \forall t$, and the system is ~~bounded~~ STABLE

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Time Invariance (a.k.a. "shift invariance," "Translation invariance")

- When talking about this property we can save time and writing by using operator notation:

$$\text{write: } y(t) = H\{x(t)\}$$

read: $y(t)$ is the output of system H when $x(t)$ is the input.

→ For a discrete-time system G with input $x[n]$ and output $y[n]$, you can similarly write

$$y[n] = G\{x[n]\}.$$

Time Invariance (continuous time):

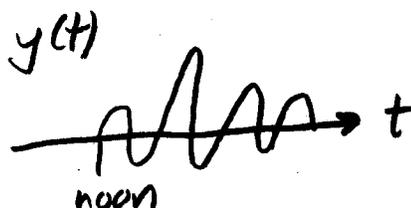
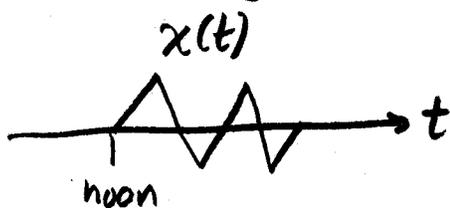
A continuous-time system H is time invariant if $\forall x(t)$ with $y(t) = H\{x(t)\}$ and $\forall t_0 \in \mathbb{R}$,
 $H\{x(t-t_0)\} = y(t-t_0)$.

Discrete-Time:

A discrete-time system H is time invariant if $\forall x[n]$ with $y[n] = H\{x[n]\}$ and $\forall n_0 \in \mathbb{Z}$,
 $H\{x[n-n_0]\} = y[n-n_0]$.

- What it means:

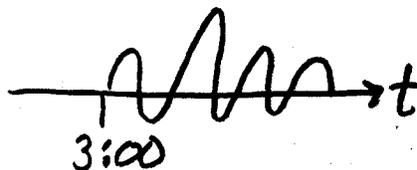
→ If I put in same signal at noon and get an output $y(t)$:



→ And then I come back at 3:00 PM and do the same thing, I get the same output as before, but starting at 3:00 PM this time:

$x(t-3\text{hours})$

$y(t-3\text{hours})$



- If the system is time invariant, then the properties of the system don't change over time.
- If a given input produces a given output today, that same input will produce the same output tomorrow. You get the same output both days in response to the same input applied both days.

EX: $y(t) = H\{x(t)\} = \cos[x(t)]$.

Let $y_1(t) = H\{x_1(t)\} = \cos[x_1(t)]$. Let $t_0 \in \mathbb{R}$.

Then $y_1(t-t_0) = \cos[x_1(t-t_0)]$.

Now let $x_2(t) = x_1(t-t_0)$.

Then $y_2(t) = H\{x_2(t)\} = \cos[x_2(t)]$
 $= \cos[x_1(t-t_0)] = y_1(t-t_0)$.

Therefore the system H is time invariant.

EX: $y[n] = \sum_{k=n-2}^{n+2} x[k]$.

Let $y_1[n] = H\{x_1[n]\} = \sum_{k=n-2}^{n+2} x_1[k]$. Let $n_0 \in \mathbb{Z}$.

Then $y_1[n-n_0] = \sum_{k=n-n_0-2}^{n-n_0+2} x_1[k]$ $\left\{ \begin{array}{l} \text{let } m = k + n_0 \\ \rightarrow k = m - n_0 \\ k = n - n_0 - 2 \rightarrow m = n - 2 \\ k = n - n_0 + 2 \rightarrow m = n + 2 \end{array} \right.$

$= \sum_{m=n-2}^{n+2} x_1[m-n_0]$ $\left\{ \begin{array}{l} \text{write "k" again instead of "m"} \end{array} \right.$

$= \sum_{k=n-2}^{n+2} x_1[k-n_0]$

Now let $x_2[n] = x_1[n-n_0]$.

Then $y_2[n] = H\{x_2[n]\} = \sum_{k=n-2}^{n+2} x_2[k]$
 $= \sum_{k=n-2}^{n+2} x_1[k-n_0] = y_1[n-n_0]$ ✓

The system is time invariant.

EX: $y[n] = nx[n]$.

Let $x_1[n] = \delta[n]$. Then $y_1[n] = H\{x_1[n]\} = n\delta[n] = 0$.

Let $x_2[n] = x_1[n-2] = \delta[n-2]$.

Then $y_2[n] = H\{x_2[n]\} = n\delta[n-2] = 2\delta[n-2]$.

But $y_1[n-2] = 0 \neq y_2[n]$, so this system is not time invariant.

NOTE: In this case, the system multiplies the input by a factor that depends on time. Thus, the properties of the system are time varying.

The proof above uses a counterexample. Here's how you could alternatively do this one using a direct proof:

Given I/O relation: $y[n] = nx[n]$.

Let $y_1[n] = H\{x_1[n]\} = nx_1[n]$ and let $n_0 \in \mathbb{Z}$.

Then $y_1[n-n_0] = (n-n_0)x_1[n-n_0]$.

Now let $x_2[n] = x_1[n-n_0]$.

Then $y_2[n] = H\{x_2[n]\} = nx_2[n] = nx_1[n-n_0]$.

Since $y_2[n] \neq y_1[n-n_0]$ in general, this system is not time invariant.

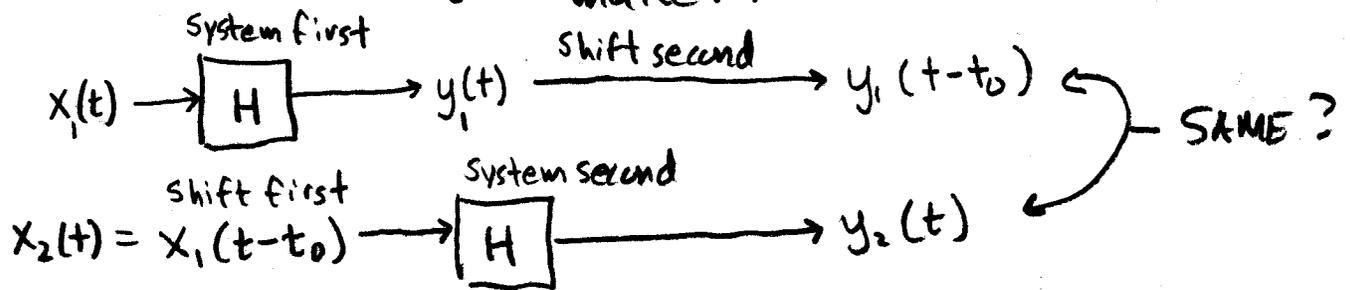
- Shift Invariance can get confusing, as you will see in the homework!

- There are two "transformations" involved:

A) shifting a signal

B) Putting a signal through the system.

- For a shift invariant system, the order of these two transformations doesn't matter:



- In answering this, the rule of thumb is:

\Rightarrow Plug in the second transformation for the independent variable in the transformation that was done first.

EX: I/O relation for system is $y(t) = x(2t)$.

Is the system shift invariant?

① Put $x_1(t)$ through the system to get $y_1(t)$, then shift to get $y_1(t-t_0)$:

1st xform: $x_1(t) \mapsto y_1(t) = x_1(2t)$

2nd xform: $t \mapsto t-t_0$

$$y_1(t-t_0) = x_1(2t) \Big|_{t=t-t_0} = x_1(2t-2t_0)$$

rule of thumb



② Shift $x_1(t)$ first to make $x_2(t) = x_1(t - t_0)$, then put $x_2(t)$ through the system to get $y_2(t)$:

1st x form: $t \mapsto t - t_0$

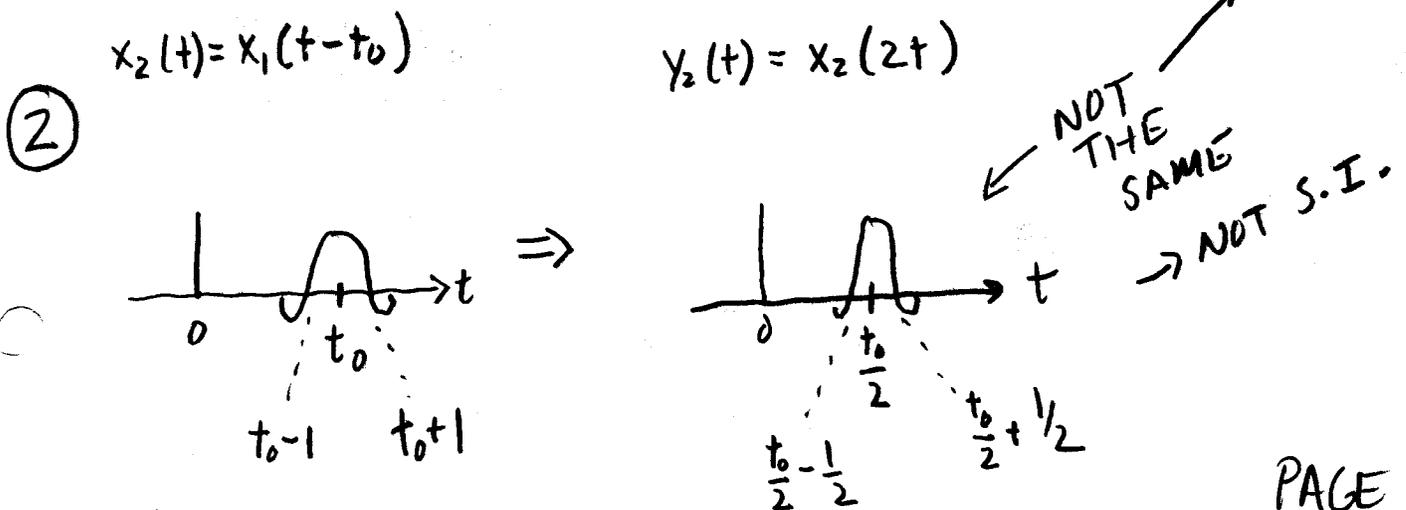
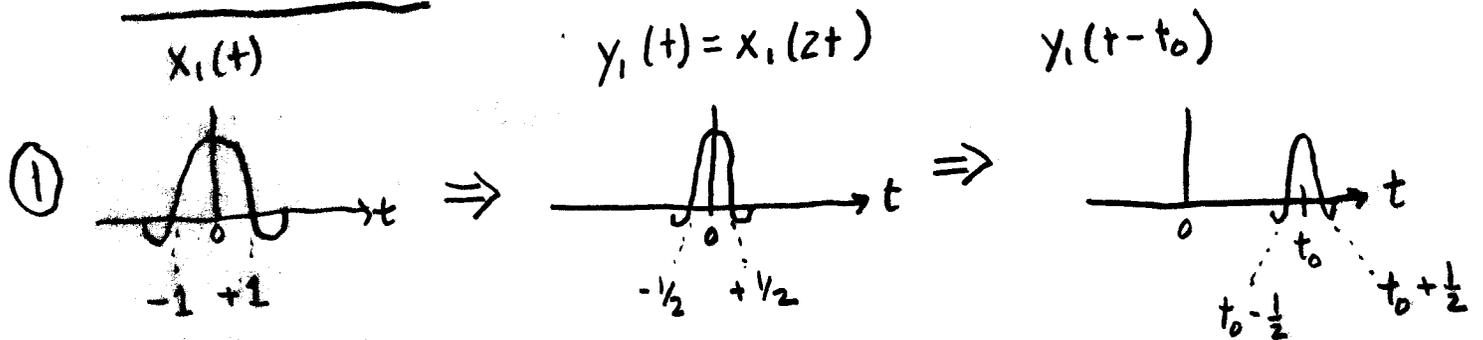
2nd x form: $x_2(t) \mapsto x_2(2t)$

Rule of thumb: plug in "2t" for "t" in first x form: this makes $2t - t_0$

So $y_2(t) = x_2(2t) = x_1(2t - t_0)$

\Rightarrow Since $y_2(t) \neq y_1(t - t_0)$, the system is NOT SHIFT INVARIANT.

Intuitively:



LINEARITY

- A continuous-time system H is linear if

$$\forall x_1(t), x_2(t) \text{ with } y_1(t) = H\{x_1(t)\} \\ \text{and } y_2(t) = H\{x_2(t)\},$$

and \forall constants $c_1, c_2 \in \mathbb{C}$, we have

$$H\{c_1 x_1(t) + c_2 x_2(t)\} = c_1 y_1(t) + c_2 y_2(t).$$

\Rightarrow In other words, for a linear system, the action of the system commutes with linear combinations.

- A discrete-time system H is linear if

$$\forall x_1[n], x_2[n] \text{ with } y_1[n] = H\{x_1[n]\} \\ \text{and } y_2[n] = H\{x_2[n]\}, \text{ and } \forall \text{ constants} \\ c_1, c_2 \in \mathbb{C},$$

$$H\{c_1 x_1[n] + c_2 x_2[n]\} = c_1 y_1[n] + c_2 y_2[n].$$

EX: $y(t) = 5$.

Let $x_1(t)$ and $x_2(t)$ be arbitrary signals and let a_1 and a_2 be arbitrary constants.

Then $y_1(t) = H\{x_1(t)\} = 5$ and $y_2(t) = H\{x_2(t)\} = 5$.

Then $a_1 y_1(t) + a_2 y_2(t) = 5(a_1 + a_2)$.

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Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$. Then

$y_3(t) = H\{x_3(t)\} = 5 \neq a_1 y_1(t) + a_2 y_2(t)$ in general \Rightarrow NOT LINEAR

NOTE: A system that is not linear is called nonlinear.

EX: $y(t) = 0$ (the "trivial" system).

Let $x_1(t)$ and $x_2(t)$ be arbitrary input signals and let a_1 and a_2 be arbitrary constants.

Then $y_1(t) = H\{x_1(t)\} = 0$ and $y_2(t) = H\{x_2(t)\} = 0$.

Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$.

Then $y_3(t) = H\{x_3(t)\} = 0 = a_1 y_1(t) + a_2 y_2(t)$.

Therefore this system is linear.

NOTE: This is the only "constant output" system that is linear.

EX: $y(t) = \frac{d}{dt} x(t)$.

Let $x_1(t)$ and $x_2(t)$ be arbitrary inputs and let a_1 and a_2 be arbitrary constants. Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$.

Then $y_3(t) = H\{x_3(t)\} = \frac{d}{dt} x_3(t)$

$$= \frac{d}{dt} [a_1 x_1(t) + a_2 x_2(t)]$$

$$= a_1 \frac{d}{dt} x_1(t) + a_2 \frac{d}{dt} x_2(t)$$

$$= a_1 H\{x_1(t)\} + a_2 H\{x_2(t)\} = a_1 y_1(t) + a_2 y_2(t) \checkmark$$

The system is linear.

EX: $y[n] = (x[n])^2$.

Let $y_1[n] = H\{x_1[n]\} = (x_1[n])^2$
and $y_2[n] = H\{x_2[n]\} = (x_2[n])^2$.

Let $x_3[n] = a_1 x_1[n] + a_2 x_2[n]$ where $a_1, a_2 \in \mathbb{C}$ are constants.

Then $y_3[n] = H\{x_3[n]\} = (x_3[n])^2$
 $= (a_1 x_1[n] + a_2 x_2[n])^2$
 $= a_1^2 x_1^2[n] + a_2^2 x_2^2[n] + 2a_1 a_2 x_1[n] x_2[n]$
 $= a_1 \{a_1 y_1[n]\} + a_2 \{a_2 y_2[n]\} + 2a_1 a_2 x_1[n] x_2[n]$
 $\neq a_1 y_1[n] + a_2 y_2[n]$ in general.

Therefore this system is nonlinear.

- The class of systems that are both linear and shift invariant is extremely important in engineering, math, and science.

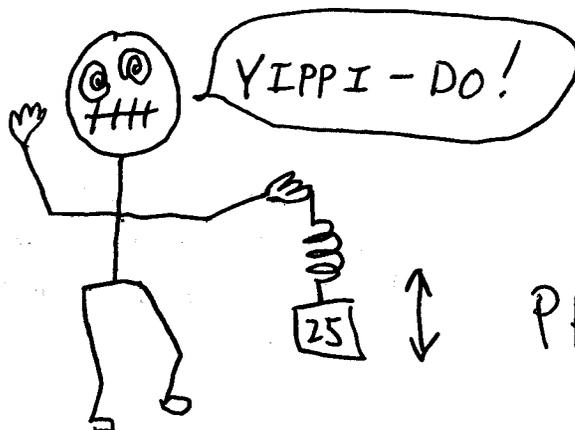
- These systems are called "Linear shift Invariant Systems," or "LSI systems!"

→ Also known as "Linear time invariant," "Linear translation invariant," or "LTI" systems.

- ECE 3793 is all about LSI systems.



- Finally, we are done with chapter One!!



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