

Chapter 2

- Recall:

- it's easy to think about vectors $\vec{v} = [v_1, v_2, v_3]^T$ in \mathbb{R}^3 .

- You can write any such \vec{v} as a linear combination of the basis $\{\vec{i}, \vec{j}, \vec{k}\}$ using dot products:

$$\begin{aligned}\vec{v} &= \langle \vec{v}, \vec{i} \rangle \vec{i} + \langle \vec{v}, \vec{j} \rangle \vec{j} + \langle \vec{v}, \vec{k} \rangle \vec{k} \\ &= v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}\end{aligned}$$

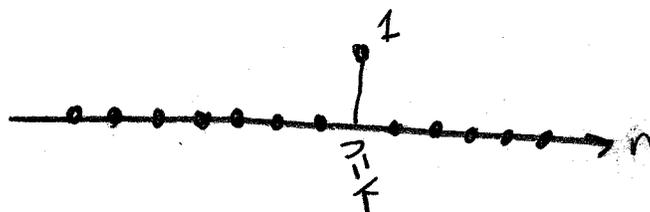
- Our discrete-time signals $x[n]$ are like vectors \vec{v} in a much larger space.

- For this space, the basis

$$\left\{ \delta[n-k] \right\}_{k \in \mathbb{Z}}$$

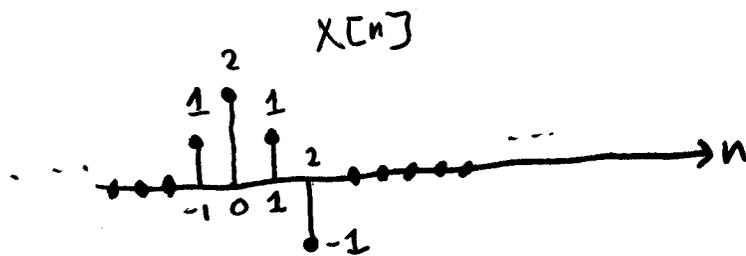
plays the same role as $\{\vec{i}, \vec{j}, \vec{k}\}$ plays in \mathbb{R}^3 :

→ Each basis signal $\delta[n-k]$ is turned on (=1) at one place and zero elsewhere:



- So we can definitely write any $x[n]$ as a linear combination of this basis by using dot products.

EX:



- Obviously, $x[n] = 1\delta[n+1] + 2\delta[n] + 1\delta[n-1] - 1\delta[n-2]$.

- Using dot products:

$$\langle x[n], \delta[n+1] \rangle = \sum_{n=-\infty}^{\infty} x[n] \delta[n+1] = x[-1] = 1$$

$$\langle x[n], \delta[n] \rangle = \sum_{n=-\infty}^{\infty} x[n] \delta[n] = x[0] = 2$$

$$\langle x[n], \delta[n-1] \rangle = \sum_{n=-\infty}^{\infty} x[n] \delta[n-1] = x[1] = 1$$

$$\langle x[n], \delta[n-2] \rangle = \sum_{n=-\infty}^{\infty} x[n] \delta[n-2] = x[2] = -1$$

($\langle x[n], \delta[n-k] \rangle = 0$ for all the other $k \in \mathbb{Z}$ for this example).

\Rightarrow This procedure always works for any $x[n]$.

- Let $x[n]$ be arbitrary.

- There are two steps to write $x[n]$ as a linear combination of the basis $\{\delta[n-k]\}_{k \in \mathbb{Z}}$.

① Find all the dot products $\langle x[n], \delta[n-k] \rangle \forall k \in \mathbb{Z}$.

② Add up the dot products times the corresponding basis signals to get $x[n]$.

Step 1: find the dot products.

- For any $k \in \mathbb{Z}$, the corresponding basis signal is $\delta[n-k]$.
- To get the coefficient for this basis signal, take the dot product

$$\langle x[n], \delta[n-k] \rangle = \sum_{n=-\infty}^{\infty} x[n] \delta[n-k]$$

(only one nonzero term, when $n=k$)

$$= x[k] \cdot \delta[0]$$

$$= \underline{\underline{x[k]}}$$

Step 2: Add up dot products times basis signals.

$$x[n] = \sum_{\text{over basis}} (k^{\text{th}} \text{ dot product}) (k^{\text{th}} \text{ basis signal})$$

$$= \sum_{k=-\infty}^{\infty} \langle x[n], \delta[n-k] \rangle \delta[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \quad \star \star \star$$

- "n" is the independent variable in $x[n]$.

- $x[n]$ is specified as a sum

- In each term of the sum, $x[k]$ is a number

- $\delta[n-k]$ is a signal with independent variable "n".

\Rightarrow k is just a "dummy" or "sum" variable. PAGE 2.3

- Now consider an LTI (LSI) system H :

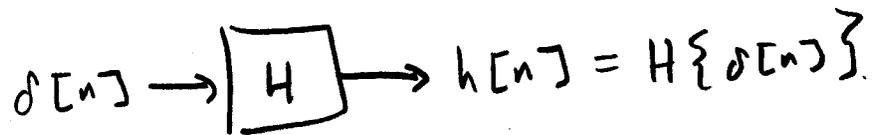


- We use "operator notation" to write the shorthand

$$y[n] = H \{ x[n] \}.$$

- Suppose the system input is $x[n] = \delta[n]$.

- In this case, we call the output the "system impulse response" $h[n]$:



\Rightarrow Because the system is shift invariant, we have

$$H \{ \delta[n-m] \} = h[n-m]$$

$$H \{ \delta[n-k] \} = h[n-k]$$

$$\forall m, k \in \mathbb{Z}.$$

\Rightarrow Because the system is Linear, we have for any constants a, b that:

$$H \{ a\delta[n-m] + b\delta[n-k] \} = ah[n-m] + bh[n-k].$$



NOTE: This is only true if H is both linear and shift invariant. Otherwise, totally different signals might come out if you put in $\delta[n]$ and $\delta[n-k]$.

EX: On page 2.2 we had the signal

$$x[n] = 1\delta[n+1] + 2\delta[n] + 1\delta[n-1] - 1\delta[n-2]$$

- If this signal is input to our LSI system H , the corresponding output is

$$\begin{aligned} y[n] &= H\{x[n]\} \\ &= H\{1\delta[n+1] + 2\delta[n] + 1\delta[n-1] - 1\delta[n-2]\} \\ &= 1H\{\delta[n+1]\} + 2H\{\delta[n]\} + 1H\{\delta[n-1]\} - 1H\{\delta[n-2]\} \\ &= 1h[n+1] + 2h[n] + 1h[n-1] - 1h[n-2] \end{aligned}$$

\Rightarrow So we see that the system impulse response $h[n]$ tells us everything there is to know about an LSI system H ... if we know $h[n]$, we can find the system output corresponding to any given input.

- This is why it is so useful to write any signal $x[n]$ as a linear combination of the basis $\{\delta[n-k]\}_{k \in \mathbb{Z}}$.

- Let's take an arbitrary input signal $x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$

and see what comes out of an LSI system H when $x[n]$ is the input:

$$y[n] = H \{ x[n] \}$$

$$= H \left\{ \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \right\}$$

↑
numbers!

$$= \sum_{k=-\infty}^{\infty} x[k] H \{ \delta[n-k] \}$$
 because H is linear

$$= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \quad \star$$
 because H is shift invariance

- So for any input $x[n]$ to any LSI system H with impulse response $h[n]$, the output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

- This is called "discrete-time convolution" and written with a " \star ":

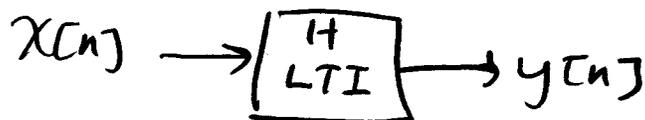
$$y[n] = x[n] \star h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

↑
a signal; a function of "n"

↑
"n" is the independent variable,
"k" is a "dummy" or "sum" variable.

Overcoming Fear of the capital sigma:

- Let $x[n]$ be a signal
- Let H be an LTI system with impulse response $h[n]$.
- write $x[n]$ as a linear combination of the natural basis and see what the output $y[n]$ looks like.
- We use two columns. They are line for line identical, but we use the " Σ " shorthand notation in the right column.

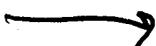


Basis:

$$\{\dots, \delta[n-1], \delta[n], \delta[n+1], \dots\}$$

Input:

$$\begin{aligned} x[n] = & \dots + x[-2]\delta[n+2] \\ & + x[-1]\delta[n+1] + x[0]\delta[n] \\ & + x[1]\delta[n-1] + x[2]\delta[n+2] + \dots \end{aligned}$$

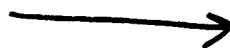


Basis:

$$\{\delta[n-k]\}_{k \in \mathbb{Z}}$$

Input:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



output:

$$\begin{aligned} y[n] &= H\{x[n]\} \\ &= H\{\dots + x[-2]\delta[n+2] + x[-1]\delta[n+1] \\ &\quad + x[0]\delta[n] + x[1]\delta[n-1] \\ &\quad + x[2]\delta[n-2] + \dots\} \end{aligned}$$

$$\begin{aligned} &= \dots + H\{x[-2]\delta[n+2]\} + H\{x[-1]\delta[n+1]\} \\ &\quad + H\{x[0]\delta[n]\} + H\{x[1]\delta[n-1]\} \\ &\quad + H\{x[2]\delta[n-2]\} + \dots \end{aligned}$$

$$\begin{aligned} &= \dots + x[-2]H\{\delta[n+2]\} + x[-1]H\{\delta[n+1]\} \\ &\quad + x[0]H\{\delta[n]\} + x[1]H\{\delta[n-1]\} \\ &\quad + x[2]H\{\delta[n-2]\} + \dots \end{aligned}$$

$$\begin{aligned} &= \dots + x[-2]h[n+2] + x[-1]h[n+1] \\ &\quad + x[0]h[n] + x[1]h[n-1] \\ &\quad + x[2]h[n-2] + \dots \end{aligned}$$

output:

$$\begin{aligned} y[n] &= H\{x[n]\} \\ &= H\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \end{aligned}$$

$$= \sum_{k=-\infty}^{\infty} H\{x[k]\delta[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]H\{\delta[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

FACT: Convolution is commutative.

- In math: $x[n] * h[n] = h[n] * x[n]$

or $\sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$

- In words: When you compute the signal

$$y[n] = x[n] * h[n],$$

the convolution of $x[n]$ and $h[n]$, it doesn't matter which signal gets the "k" and which signal gets the "n-k" in the convolution sum.

\Rightarrow For every "n", $y[n]$ is the exact same number either way you do it.

Proof: Let $y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$.

Let $y_2[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$.

Then $y_1[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$

$\left. \begin{array}{l} m=n-k \\ k=n-m \end{array} \right\} = \sum_{m=-\infty}^{\infty} x[n-m] h[m]$

(order of summation doesn't matter) $= \sum_{m=-\infty}^{\infty} x[n-m] h[m]$

$$= \sum_{m=-\infty}^{\infty} h[m] x[n-m]$$

(write "k" instead of "m")

$$= \sum_{k=-\infty}^{\infty} h[k] x[n-k] = y_2[n] \checkmark$$

- We have shown that the output of a discrete-time LSI system is the convolution of the input with the impulse response.

NOTE: when we say "H is a system with impulse response $h[n]$ ", we imply that H is an LSI system.

(if H is not LSI, then "impulse response" has no meaning)

- We have shown that, in computing a convolution, it doesn't matter which signal gets the "k" and which signal gets the "n-k":

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} h[k] x[n-k].$$

NOTE: The operation of convolving two signals can be defined even when we aren't talking about a system.

For example, define

$$\begin{aligned} x_3[n] &= x_1[n] * x_2[n] \\ &= \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \\ &= \sum_{k=-\infty}^{\infty} x_1[n-k] x_2[k] \end{aligned}$$

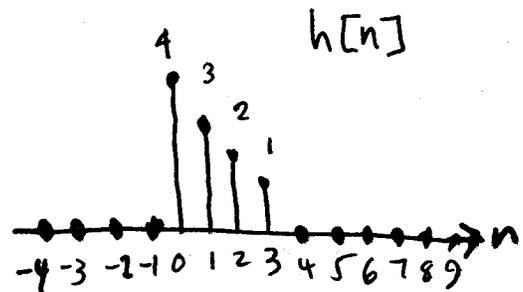
Two ways of writing the same signal. The "k" sums away leaving $x_3[n]$, a function of "n".

- We have used easy math to show analytically that the output of any discrete-time LSI system is the convolution of the input with the unit pulse response.

- Now let's develop intuition to see why this must be true.

- Suppose H is a discrete-time LSI system with unit pulse response

$$h[n] = \begin{cases} 0, & n < 0 \\ 4, & n = 0 \\ 3, & n = 1 \\ 2, & n = 2 \\ 1, & n = 3 \\ 0, & n > 3 \end{cases} =$$

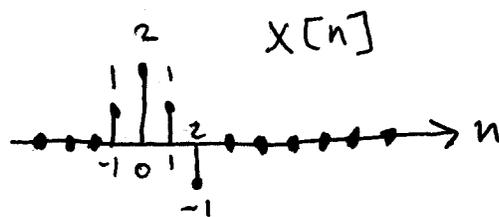


NOTE: By taking inner products, you can see that

$$h[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + 1\delta[n-3].$$

- Let the input be our sample $x[n]$ from page 2.2:

$$x[n] = 1\delta[n+1] + 2\delta[n] + 1\delta[n-1] - 1\delta[n-2]$$



- We know that

$$y[n] = H\{x[n]\}$$

$$= H\{1\delta[n+1]\} + H\{2\delta[n]\} + H\{1\delta[n-1]\} + H\{-1\delta[n-2]\}$$

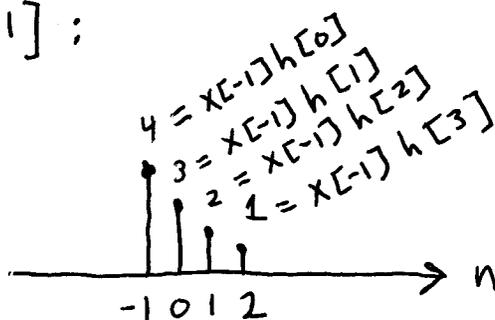
response due
to input $x[-1]$

response due
to input $x[0]$

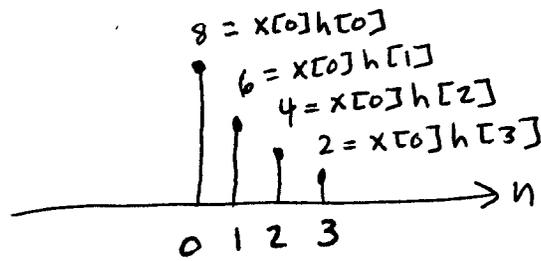
response due
to input $x[1]$

response due
to input $x[2]$

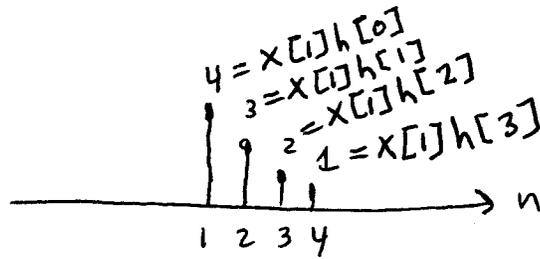
- For the input at time $n = -1$, $1\delta[n+1]$, we get a response $1h[n+1]$:



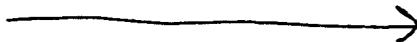
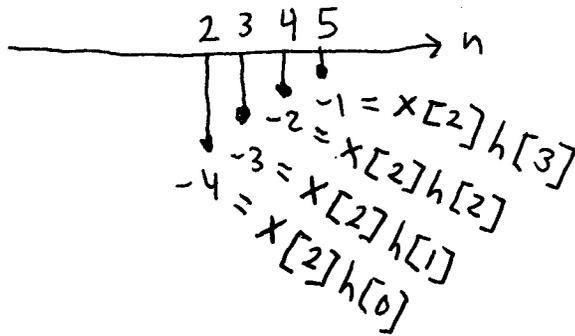
- For the input at $n=0$ ($2\delta[n]$), we get a response $2h[n]$:



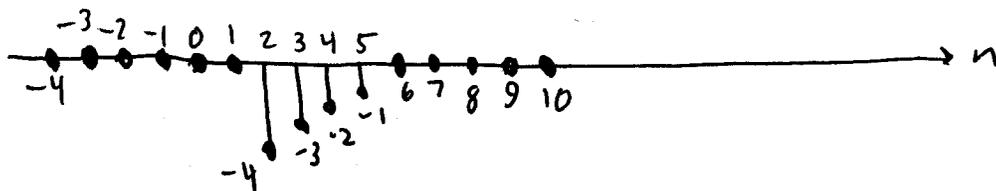
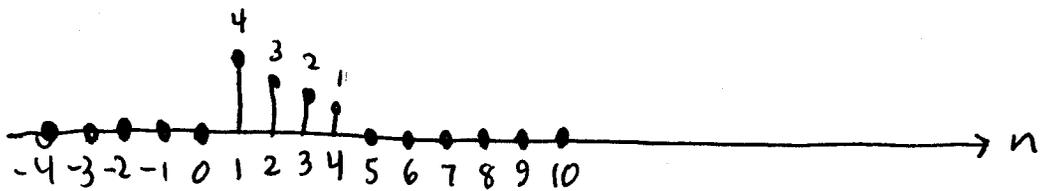
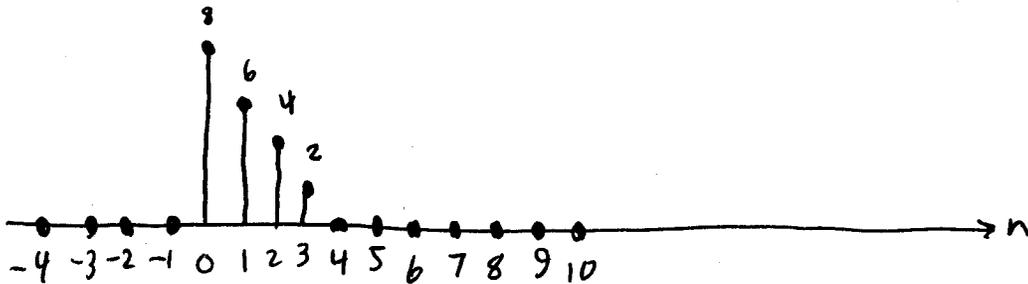
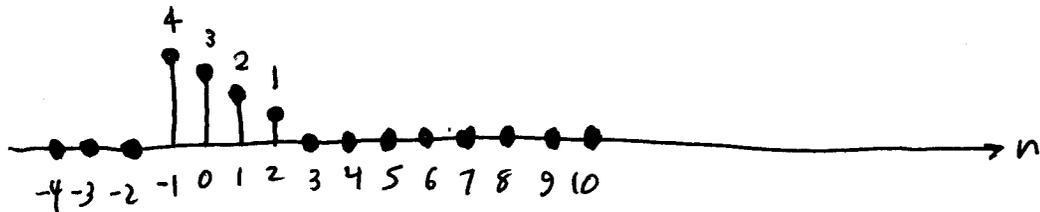
- For the input at $n=1$ ($1\delta[n-1]$), we get a response $1h[n-1]$:



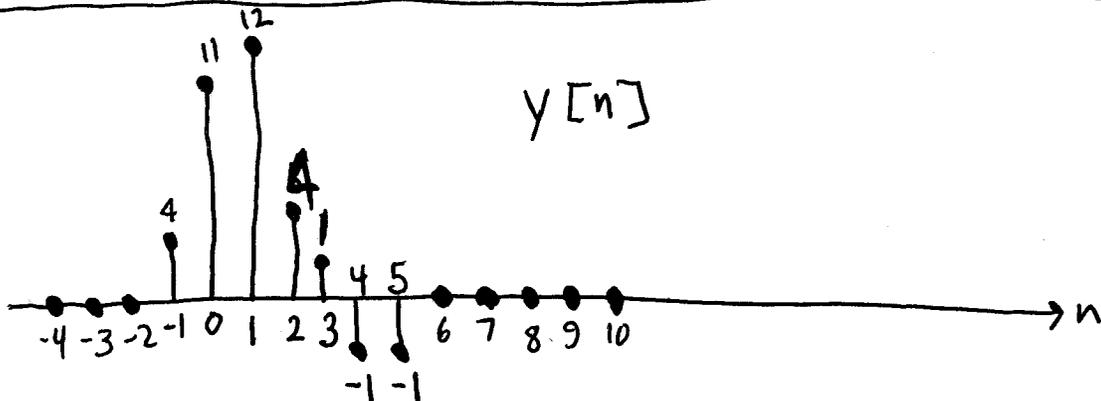
- For the input at $n=2$, ($-1\delta[n-2]$), we get a response $-1h[n-2]$:



- The total response $y[n]$ is the sum of these individual responses:



+



- So,

$$y[n] = 4\delta[n+1] + 11\delta[n] + 12\delta[n-1] + 4\delta[n-2] + \delta[n-3] \\ - \delta[n-4] - \delta[n-5].$$

(by taking inner products). - from graph

- One way to think of this is that:

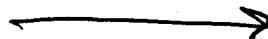
Input at $n=-1$ is $1\delta[n+1] \rightarrow$ Causes $1 \cdot h[n]$ to start coming out at $n=-1$.

Input at $n=0$ is $2\delta[n] \rightarrow$ Causes $2h[n]$ to start coming out at $n=0$.

Input at $n=1$ is $1\delta[n-1] \rightarrow$ Causes $1h[n]$ to start coming out at $n=1$.

Input at $n=2$ is $-1\delta[n-2] \rightarrow$ Causes $-h[n]$ to start coming out at $n=2$.

- We can tabulate this



Output

Input	n=-1	n=0	n=1	n=2	n=3	n=4	n=5
n=-1: X[-1] = 1	X[-1]h[0] = 4	X[-1]h[1] = 3	X[-1]h[2] = 2	X[-1]h[3] = 1	0	0	0
n=0: X[0] = 2	0	X[0]h[0] = 8	X[0]h[1] = 6	X[0]h[2] = 4	X[0]h[3] = 2	0	0
n=1: X[1] = 1	0	0	X[1]h[0] = 4	X[1]h[1] = 3	X[1]h[2] = 2	X[1]h[3] = 1	0
n=2: X[2] = -1	0	0	0	X[2]h[0] = -4	X[2]h[1] = -3	X[2]h[2] = -2	X[2]h[3] = -1

NOTE: indices of x go up, indices of h go down

At n=2,

- X[-1]h[3] is still coming out because of X[-1]
- X[0]h[2] is still coming out because of X[0]
- X[1]h[1] is still coming out because of X[1]
- X[2]h[0] is coming out because of X[2].

$$\begin{aligned}
 Y[2] &= X[-1]h[3] + X[0]h[2] + X[1]h[1] + X[2]h[0] \\
 &= \sum_{k=-\infty}^{\infty} X[k]h[2-k]
 \end{aligned}$$

- On the last page, there was nothing special about "2".

- Write "n" instead of "2":

$$y[n] = x[-1]h[n-(-1)] + x[0]h[n-0] + x[1]h[n-1] + x[2]h[n-2]$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

recall: $x[n] = 0$ if $n < -1$ or $n > 3$.
Also, $h[n] = 0$ if $n < 0$ or $n > 3$.

⇒ The "other way"

- From the table,

$$y[2] = x[-1]h[3] + x[0]h[2] + x[1]h[1] + x[2]h[0]$$

$$= h[0]x[2] + h[1]x[1] + h[2]x[0] + h[3]x[-1]$$

$$= \sum_{k=0}^3 h[k]x[2-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]x[2-k], \quad \text{since } h[n] = 0, n \notin [0, 3].$$

- And in general,

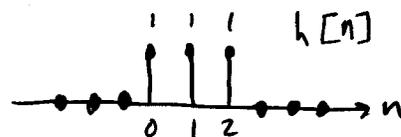
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

2.15

EX: graphical convolution.

H is an LSI system with unit pulse response

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

Note: For any fixed n , say $n=5$, this is an inner

product:

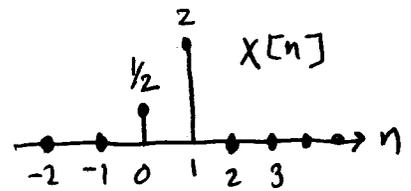
$$y[5] = \sum_{k=-\infty}^{\infty} x[k] h[-k - (-5)]$$
$$= \langle x[k], h[-k - (-5)] \rangle.$$

→ Thus, for any fixed n , we can find $y[n]$ by multiplying the corresponding values of $x[k]$ and $h[-k - (-n)]$ and adding up the products.

→ Doing this graphically for each n is called "graphical convolution!"



- Suppose $x[n] = \frac{1}{2}\delta[n] + 2\delta[n-1]$

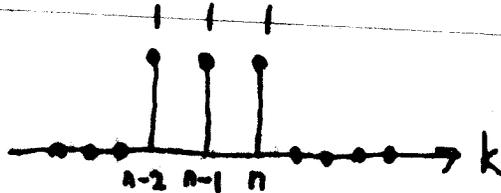


Graph of $h[k - (-n)]$:

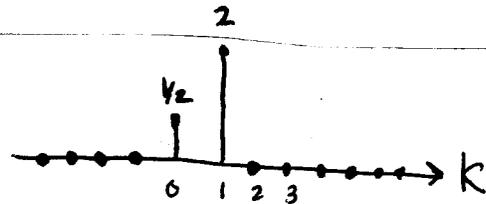


NOTE :
independent
variable is k.

Graph of $h[-k - (-n)] = h[n - k]$:



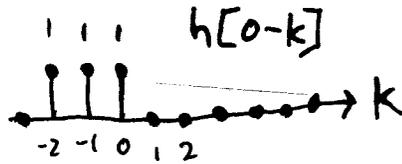
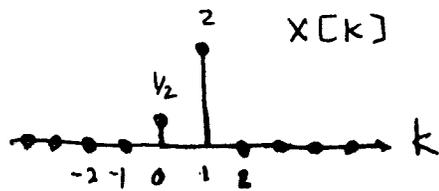
Graph of $x[k]$:



- For $n < 0$, all the products $x[k]h[n-k]$ are zero,
so $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = 0, n < 0$

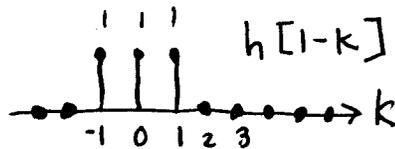
- For $n-2 > 1$, or $n > 3$, $x[k]h[n-k]$ is also
zero, so $y[n] = 0, n > 3$.

For $n=0$,



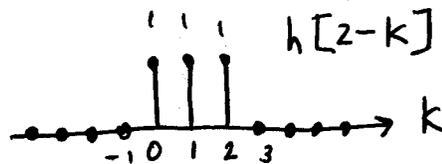
$$y[0] = 0 \cdot 1 + 0 \cdot 1 + \frac{1}{2} \cdot 1 = \frac{1}{2}$$

For $n=1$,



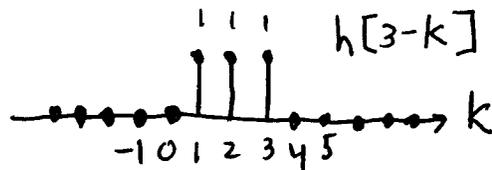
$$y[1] = 0 \cdot 1 + \frac{1}{2} \cdot 1 + 2 \cdot 1 = \frac{5}{2}$$

For $n=2$,



$$y[2] = \frac{1}{2} \cdot 1 + 2 \cdot 1 + 0 \cdot 1 = \frac{5}{2}$$

For $n=3$,

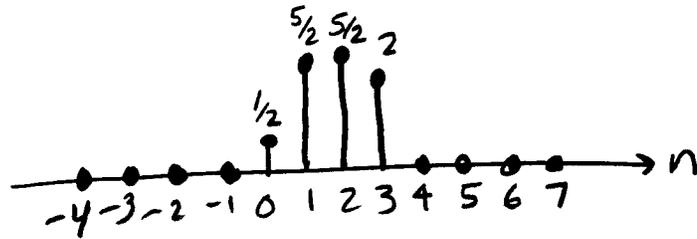


$$y[3] = \frac{1}{2} \cdot 0 + 1 \cdot 2 + 0 \cdot 1 = 2$$

- So,

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1}{2}, & n = 0 \\ \frac{5}{2}, & 1 \leq n \leq 2 \\ 2, & n = 3 \\ 0, & n > 3 \end{cases}$$

$$- \text{So, } y[n] = \frac{1}{2}\delta[n] + \frac{5}{2}\delta[n-1] + \frac{5}{2}\delta[n-2] + 2\delta[n-3]$$



☆☆ How to work the problems on a test without making mistakes:

① Choose which signal gets "k" and which one gets "n-k", i.e.,

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \text{OR} \quad \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

→ You can always get the right answer either way, but one way might be slightly easier to work out.

→ Usually it's best to pick the one with the simpler expression to get the "n-k".

→ For the following steps, assume we picked the first way, i.e., $x[k]$ and $h[n-k]$.

② Make a graph of $x[k]$.

③ Make a graph of $h[n-k]$ (as a function of k).

→ To avoid making mistakes, it's best to do this in three steps.

③A Graph $h[k]$.

③B Shift the graph in ③A to the right by $-n$ to obtain the graph of $h[k - (-n)] = h[n+k]$.

③C Flip the graph in ③B with respect to the k -axis to obtain the graph of $h[n-k]$.

④ To find $y[n]$, you must multiply the graphs in ② and ③C, and then add up the product graph from $k = -\infty$ to $k = +\infty$.

⇒ The product graph generally depends on "n".

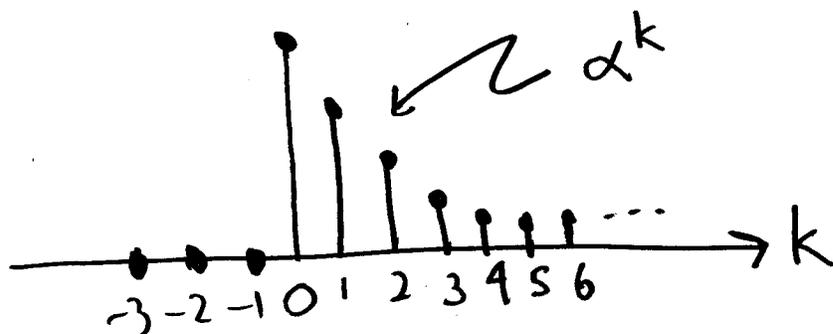
⇒ In the last example, we worked the sum for each "n".

⇒ But you hope that you can examine the graphs in (2) and (3C) and find just a few expressions that will cover all of the "n"s.

EX: $x[n] = \alpha^n u[n]$, $0 < \alpha < 1$
 $h[n] = u[n]$.

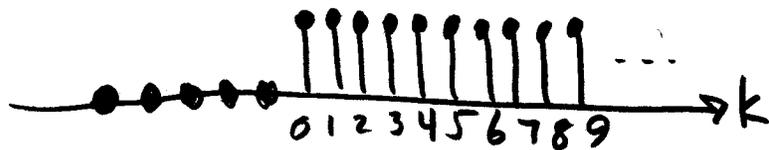
→ Since $x[n]$ has the more complicated expression, we will put the "k" on x and the "n-k" on h .

- Graph $x[k]$:



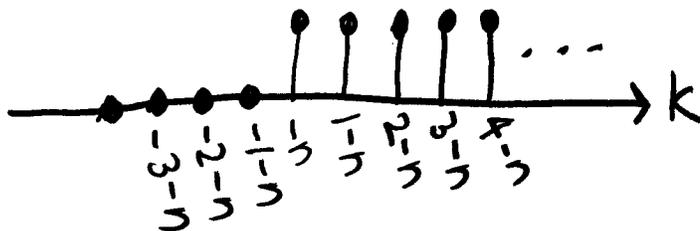
- Graph $h[n-k]$ in three steps:

$h[k]$

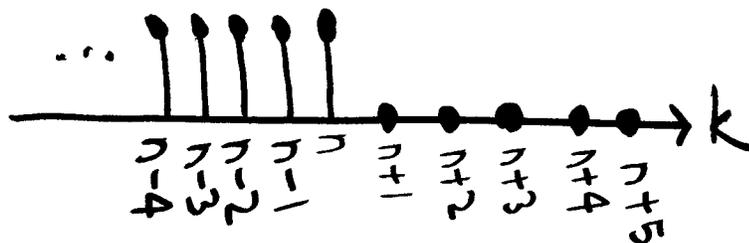


$$h[k - (-n)] = h[n+k]$$

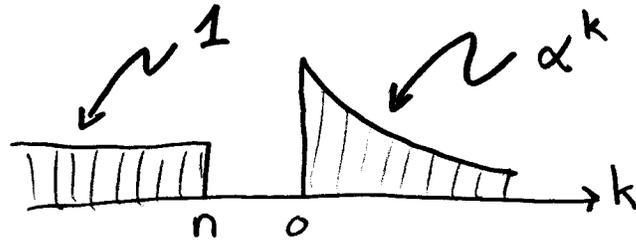
(shift right by $-n$)



$h[n-k]$ (flip)



- When $n < 0$, we have



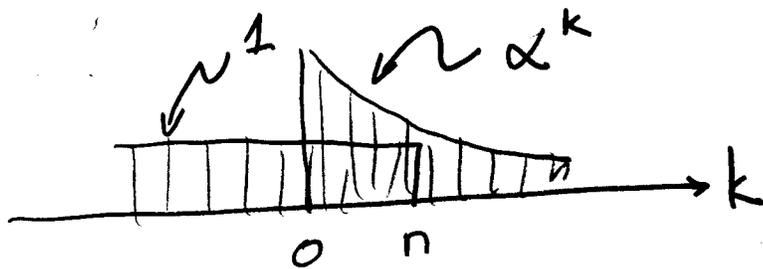
NOTE: these graphs are just intended to show where $x[k]$ and $h[n-k]$ are nonzero. It's easier and clearer to draw them like continuous-time functions even though they are really discrete.

→ From the graphs above, we see that, when $n < 0$, for every k either $x[k] = 0$ or $h[n-k] = 0$.

→ So the product $x[k]h[n-k] = 0 \quad \forall k$ when $n < 0$.

→ So $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} 0 = 0$ when $n < 0$.

- When $n \geq 0$, we have



→ In this case the product $x[k]h[n-k]$ will be nonzero from $k=0$ to $k=n$, but zero for all other k .

→ So the sum $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ has nonzero terms from $k=0$ to $k=n$.

→ In this range of k , we have
(see graphs on page 2.21, 2.22)

$$x[k] = \alpha^k \text{ and } h[n-k] = 1.$$

→ So, for $n \geq 0$, we have

$$y[n] = \sum_{k=0}^n \alpha^k \cdot 1 = \sum_{k=0}^n \alpha^k$$

Applying the sum formula $\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \alpha \neq 1$

$$\text{We have } y[n] = \frac{1 - \alpha^{n+1}}{1 - \alpha}$$

- For this problem, these two cases cover all of the "n"s.
- Putting it all together:

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1 - \alpha^{n+1}}{1 - \alpha}, & n \geq 0 \end{cases}$$

- This can also be written as

$$y[n] = \frac{1 - \alpha^{n+1}}{1 - \alpha} u[n].$$

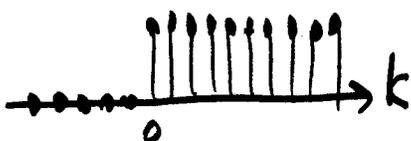
Let's try working this one the "other way."

$$x[n] = \alpha^n u[n], \quad 0 < \alpha < 1$$

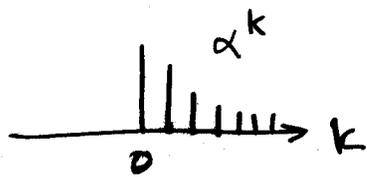
$$h[n] = u[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

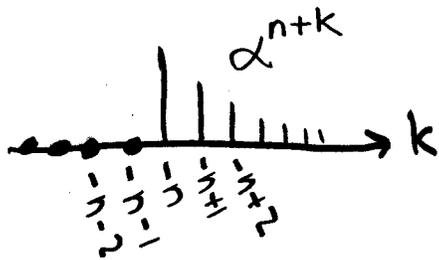
$h[k]$



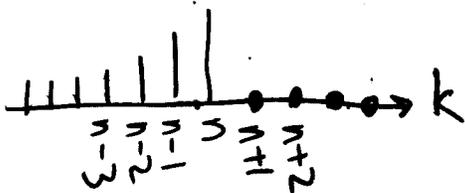
$x[k]$



$$x[k-n] = x[n+k]$$



$x[n-k]$

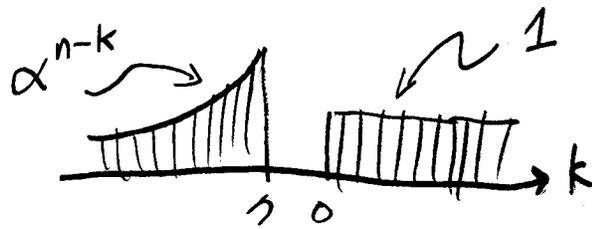


Recall from last page:

$h[k]$

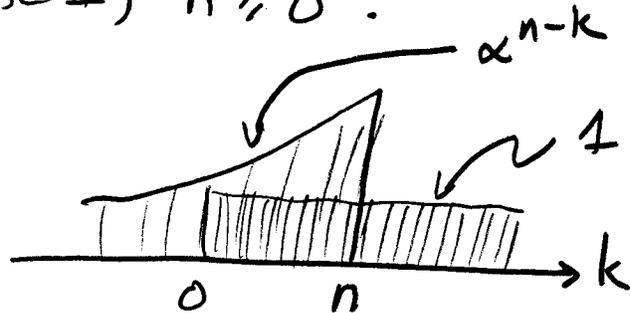


Case I) $n < 0$:



No overlap: $y[n] = 0$.

Case II) $n \geq 0$:



$$y[n] = \sum_{k=0}^n \alpha^{n-k} \cdot 1 = \sum_{k=0}^n \alpha^n \alpha^{-k}$$

$$= \alpha^n \sum_{k=0}^n \alpha^{-k} = \alpha^n \sum_{k=0}^n \left(\frac{1}{\alpha}\right)^k$$

$$= \alpha^n \frac{\left(\frac{1}{\alpha}\right)^0 - \left(\frac{1}{\alpha}\right)^{n+1}}{1 - \frac{1}{\alpha}}$$

$$= \alpha^n \frac{1 - \alpha^{-n-1}}{1 - \alpha^{-1}} \cdot \frac{\alpha}{\alpha}$$

$$= \alpha^n \frac{\alpha - \alpha^{-n}}{\alpha - 1} = \frac{\alpha^{n+1} - 1}{\alpha - 1}$$

$$= \frac{1 - \alpha^{n+1}}{1 - \alpha} \quad \checkmark$$

All Together:

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1 - \alpha^{n+1}}{1 - \alpha}, & n \geq 0 \end{cases}$$

⇒ Note that this agrees with the solution we obtained on page 2.25.

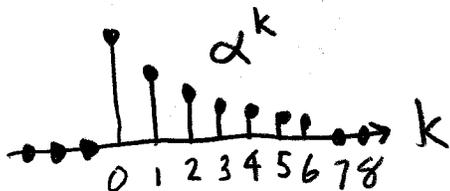
EX: $x[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases} = u[n] - u[n-5]$

$$h[n] = \begin{cases} \alpha^n, & 0 \leq n \leq 6 \\ 0, & \text{otherwise} \end{cases} = \alpha^n \{u[n] - u[n-7]\},$$

$0 < \alpha < 1$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

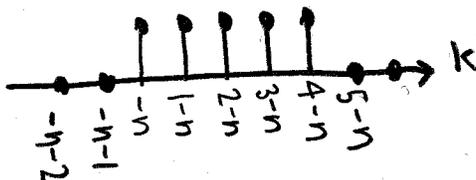
$h[k]$

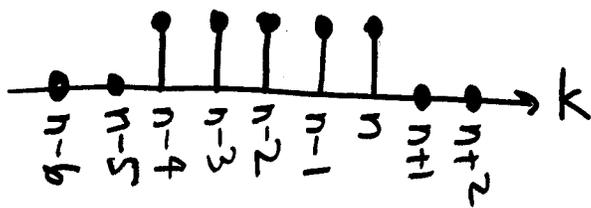
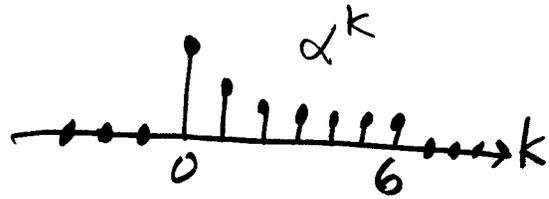


$x[k]$

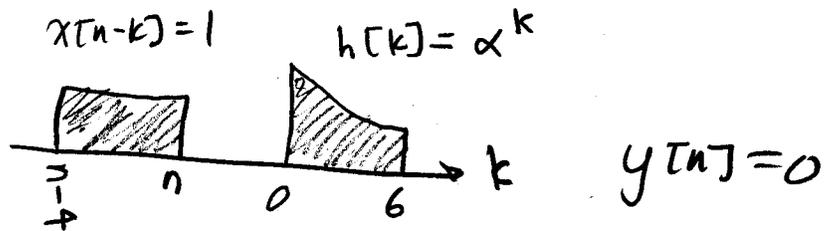


$$x[k-n] = x[n+k]$$

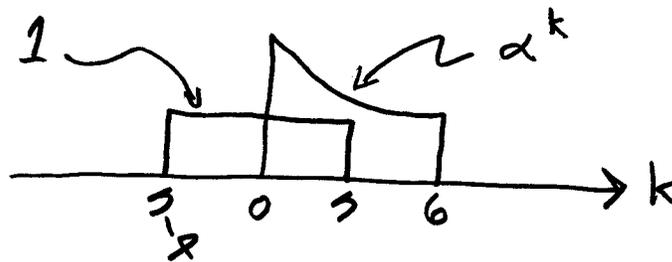


$x[n-k]$  $h[k]$ 

case I) $n < 0$:



case II) $n \geq 0$ and $n-4 < 0$: $0 \leq n < 4$:



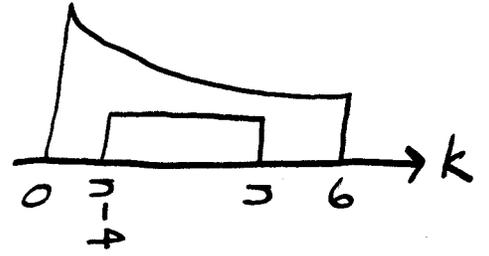
In this case, $h[k]$ "turns the sum on" at $k=0$ and $x[n-k]$ "turns it off" at $k=n$.

$$y[n] = \sum_{k=0}^n h[k] x[n-k] = \sum_{k=0}^n \alpha^k \cdot 1 = \sum_{k=0}^n \alpha^k$$

$$= \frac{\alpha^0 - \alpha^{n+1}}{1 - \alpha} = \frac{1 - \alpha^{n+1}}{1 - \alpha}$$

Case III) $n-4 \geq 0$ and $n < 6$: $4 \leq n < 6$:

In this case $x[n-k]$ turns the sum on at $k=n-4$ and also turns it off at $k=n$.

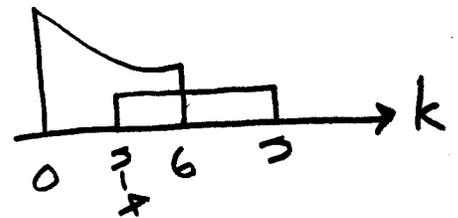


$$y[n] = \sum_{k=n-4}^n h[k] x[n-k]$$

$$= \sum_{k=n-4}^n \alpha^k = \frac{\alpha^{n-4} - \alpha^{n+1}}{1 - \alpha}$$

Case IV) $n \geq 6$ and $n-4 < 7$: $6 \leq n < 11$:

$x[n-k]$ turns sum on at $k=n-4$ and $h[k]$ turns it off at $k=6$.

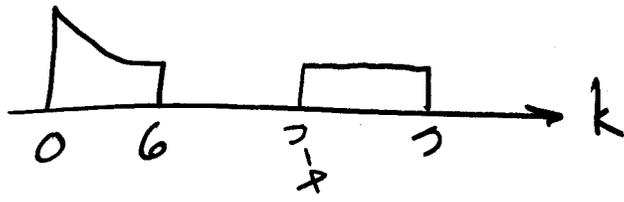


$$y[n] = \sum_{k=n-4}^6 h[k] x[n-k]$$

$$= \sum_{k=n-4}^6 \alpha^k = \frac{\alpha^{n-4} - \alpha^7}{1 - \alpha}$$

→

Case II) $n \geq 11$:



$$y[n] = 0.$$

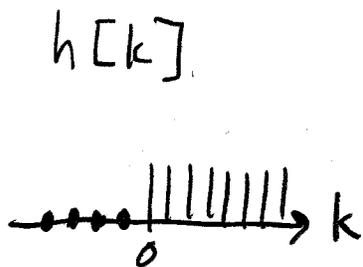
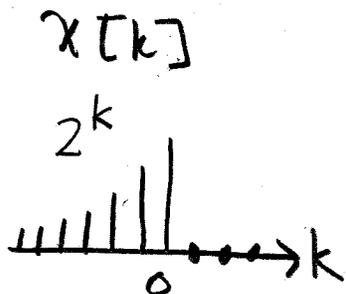
All Together:

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1 - \alpha^{n+1}}{1 - \alpha}, & 0 \leq n < 4 \\ \frac{\alpha^{n-4} - \alpha^{n+1}}{1 - \alpha}, & 4 \leq n < 6 \\ \frac{\alpha^{n-4} - \alpha^7}{1 - \alpha}, & 6 \leq n < 11 \\ 0, & n \geq 11 \end{cases}$$

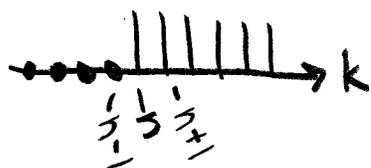
EX: $x[n] = 2^n u[-n]$

$h[n] = u[n]$

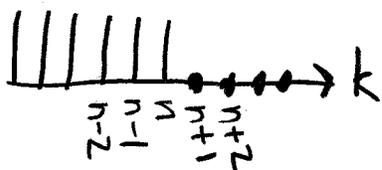
$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$



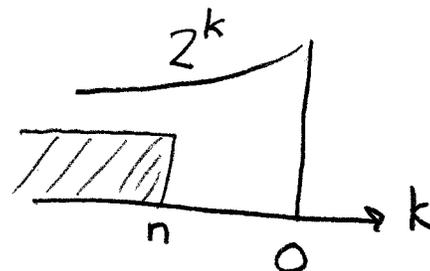
$h[k-n] = h[n+k]$



$h[n-k]$



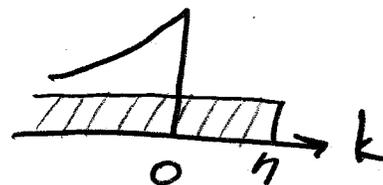
Case I) $n < 0$:



$y[n] = \sum_{k=-\infty}^n 2^k \cdot 1$

$= \lim_{A \rightarrow \infty} \sum_{k=-A}^n 2^k = \lim_{A \rightarrow \infty} \frac{2^{-A} - 2^{n+1}}{1-2}$
 $= \frac{0 - 2^{n+1}}{-1} = 2^{n+1}$

Case II) $n \geq 0$:



$y[n] = \sum_{k=-\infty}^0 2^k$

let $m = -k$; $k = -m$.

when $k \rightarrow -\infty$, $m \rightarrow \infty$

when $k = 0$, $m = 0$.

$y[n] = \sum_{m=\infty}^0 2^{-m}$

→

- But this is just adding. You can sum the terms in any order you like and it doesn't change anything.

$$y[n] = \sum_{m=0}^{\infty} 2^{-m} = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = \frac{1}{1-\frac{1}{2}}$$
$$= \frac{1}{\frac{1}{2}} = 2.$$

All Together:

$$y[n] = \begin{cases} 2^{n+1}, & n < 0 \\ 2, & n \geq 0 \end{cases}$$

Response of Continuous-Time LSI Systems

- The set $\{\delta(t-\tau)\}_{\tau \in \mathbb{R}}$ is a basis for the space of continuous-time signals $x(t)$.

- Recall definition of dot product for such signals:

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t) g^*(t) dt$$

- For each basis signal, we need to use the dot product to get a coefficient:

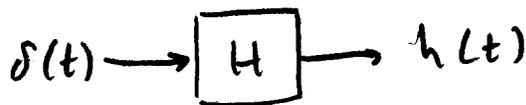
$$\langle x(t), \delta(t-\tau) \rangle = \int_{-\infty}^{\infty} x(t) \delta(t-\tau) dt = x(\tau)$$

- Now add up coefficients times basis signals:

$$\begin{aligned} x(t) &= \sum_{\text{basis}} (\text{coef}) \cdot (\text{basis signal}) \\ &= \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau \end{aligned} \quad \left. \begin{array}{l} \text{note:} \\ = \int_{-\infty}^{\infty} x(t-\tau) \delta(\tau) d\tau \end{array} \right\}$$

↑ a number as far as "t" is concerned
a signal w/ independent variable "t".

- Let H be an LTI system w/ impulse response $h(t)$:



- Then if $H\{x_1(t)\} = y_1(t)$, $H\{x_2(t)\} = y_2(t)$, and a, b are constants,

$$H\{ax_1(t) + bx_2(t)\} = ay_1(t) + by_2(t).$$

- What is the system output when the input is

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau \quad ?$$

$$y(t) = H\{x(t)\} = H\left\{\int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau\right\}$$

$$= \int_{-\infty}^{\infty} H\{x(\tau) \delta(t-\tau)\} d\tau$$

↑ a number!

) because H is linear

$$= \int_{-\infty}^{\infty} x(\tau) H\{\delta(t-\tau)\} d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \quad \text{because H is shift invariant}$$

$$\equiv x(t) * h(t)$$

a signal w/ independent variable "t".

⇒ This is the definition of convolution for continuous-time.

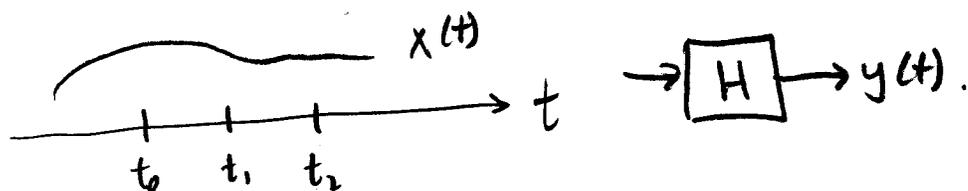
For any two signals $x_1(t)$ and $x_2(t)$, we call the signal

$$\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

the "convolution of $x_1(t)$ and $x_2(t)$ ".

⇒ We have shown that for any continuous-time LTI system, the output is the convolution of the input with the system impulse response

- The interpretation is the same as it was in discrete time:



- input term $x(t_0)\delta(t-t_0)$ arrives at time $t=t_0$ and causes $x(t_0)h(t-t_0)$ to start coming out @ $t=t_0$

- " " $x(t_1)\delta(t-t_1)$ " " " $t=t_1$ "

" $x(t_1)h(t-t_1)$ " " " " @ $t=t_1$

- " " $x(t_2)\delta(t-t_2)$ " " " $t=t_2$ "

" $x(t_2)h(t-t_2)$ " " " " @ $t=t_2$.

⇒ The total response includes all of these terms and others:

$$y(t_2) = \dots + x(t_0)h(t-t_0) + x(t_1)h(t-t_1) + x(t_2)h(t-t_2) + \dots$$

⇒ And in general, to find the total response at t_2 we must add up the individual responses due to all input terms:

$$y(t_2) = \int_{-\infty}^{\infty} x(\tau)h(t_2-\tau)d\tau$$

⇒ This is true for every time - not just t_2 . So more generally

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t).$$

NOTE: Like discrete-time convolution, continuous-time convolution is also commutative:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$= - \int_{\infty}^{-\infty} x(t-\theta) h(\theta) d\theta$$

$$= \int_{-\infty}^{\infty} h(\theta) x(t-\theta) d\theta$$

$$= h(t) * x(t).$$

$$\theta = t - \tau \quad d\theta = -d\tau$$

$$\tau = t - \theta$$

$$\tau \rightarrow -\infty \Rightarrow \theta \rightarrow \infty$$

$$\tau \rightarrow \infty \Rightarrow \theta \rightarrow -\infty$$

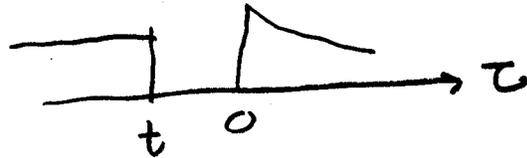
\Rightarrow They are the same number for every t .

EX: $x(t) = e^{-at} u(t), a > 0.$

$h(t) = u(t)$

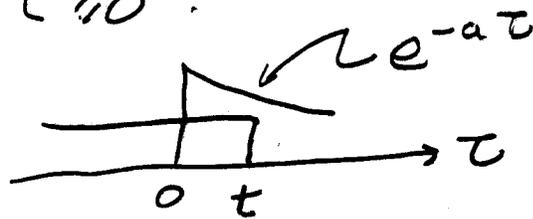
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

case I) $t < 0$:



$$y(t) = \int_{-\infty}^{\infty} 0 d\tau = 0.$$

case II) $t \geq 0$:



$$\begin{aligned} y(t) &= \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} [e^{-a\tau}]_{\tau=0}^t \\ &= -\frac{1}{a} [e^{-at} - 1] = \frac{1}{a} [1 - e^{-at}] \end{aligned}$$

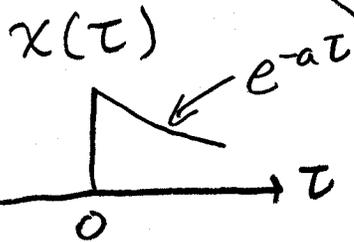
ALL TOGETHER: $y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{a} [1 - e^{-at}], & t \geq 0 \end{cases}$

$= \frac{1}{a} [1 - e^{-at}] u(t).$

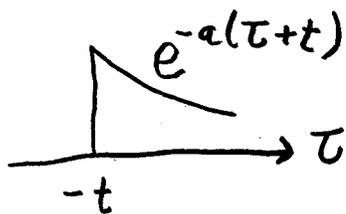
EX: Same problem the "other way".

$$x(t) = e^{-at}u(t), \quad a > 0$$

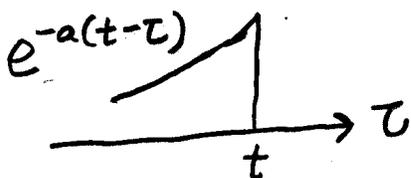
$$h(t) = u(t)$$



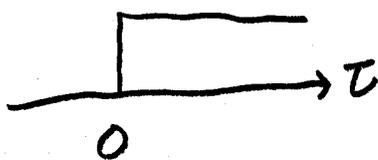
$$x(\tau - t) = x(\tau + t)$$



$$x(t - \tau)$$

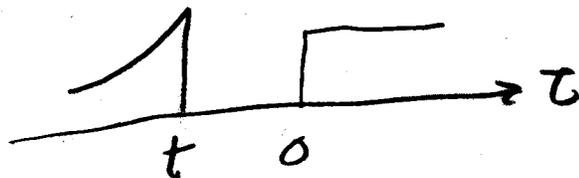


$$h(\tau)$$



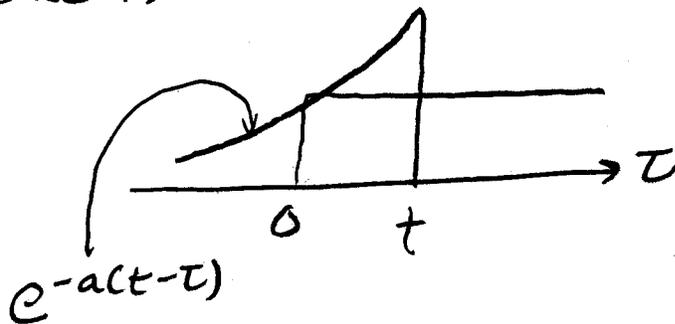
$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau$$

Case I) $t < 0$:



$$y(t) = \int_{-\infty}^{\infty} 0 d\tau = 0.$$

Case II) $t > 0$:



$$y(t) = \int_0^t e^{-a(t-\tau)} d\tau = \int_0^t e^{-at} e^{a\tau} d\tau$$

$$= e^{-at} \int_0^t e^{a\tau} d\tau$$

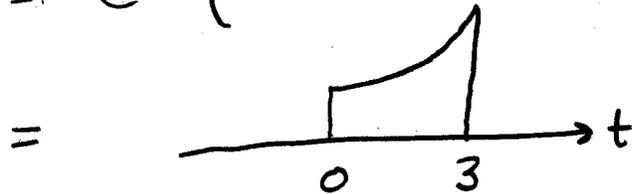
$$= \frac{1}{a} e^{-at} [e^{a\tau}]_{\tau=0}^t = \frac{1}{a} e^{-at} [e^{at} - 1]$$

$$= \frac{1}{a} [1 - e^{-at}]$$

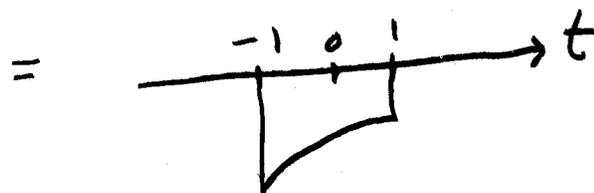
All Together: $y(t) = \frac{1}{a} [1 - e^{-at}] u(t).$

EX: $x(t) = \begin{cases} e^{3t}, & 0 \leq t \leq 3 \\ 0, & \text{other} \end{cases}$

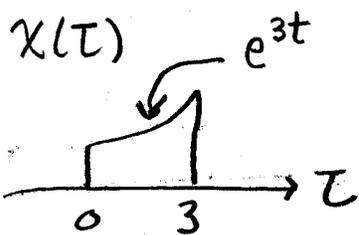
$$= e^{3t} \{ u(t) - u(t-3) \}$$



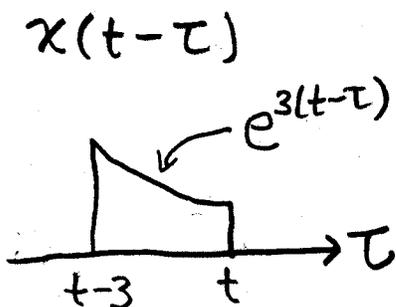
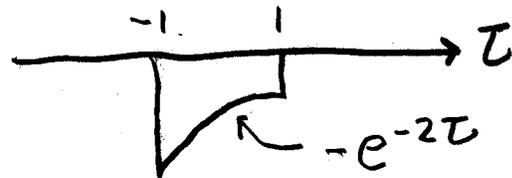
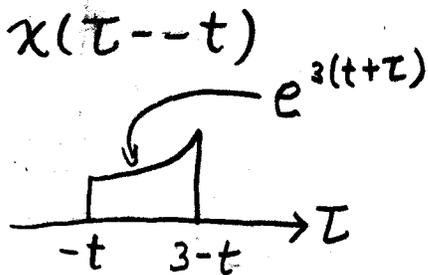
$$h(t) = \begin{cases} -e^{-2t}, & -1 \leq t \leq 1 \\ 0, & \text{other} \end{cases} = -e^{-2t} \{ u(t+1) - u(t-1) \}$$



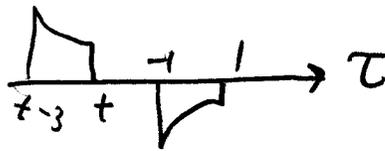
$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$



$h(\tau)$

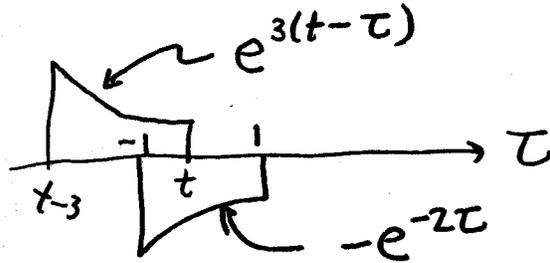


Case I) $t < -1$:



$$y(t) = \int_{-\infty}^{\infty} 0 d\tau = 0.$$

Case II) $t > -1$ and $t < 1$: $-1 \leq t < 1$:



$$y(t) = \int_{-1}^t e^{3(t-\tau)} [-e^{-2\tau}] d\tau = - \int_{-1}^t e^{3t} e^{-3\tau} e^{-2\tau} d\tau$$

$$= -e^{3t} \int_{-1}^t e^{-5\tau} d\tau = \frac{1}{5} e^{3t} [e^{-5\tau}]_{\tau=-1}^t$$

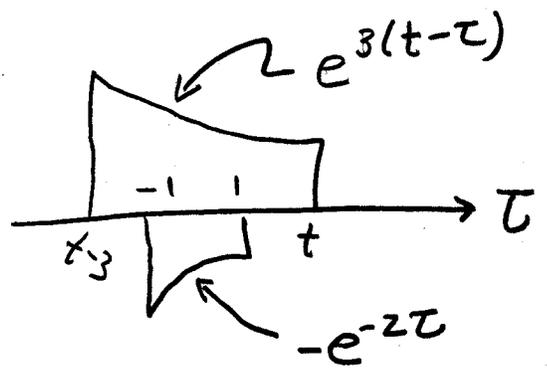
$$= \frac{1}{5} e^{3t} [e^{-5t} - e^5] = \frac{1}{5} [e^{-2t} - e^{3t+5}]$$

$$= \frac{1}{5} e^{-2t} - \frac{1}{5} e^{3t+5}$$



Case III) $t > 1$ and $t-3 < -1$
 $t > 1$ and $t < 2$

$1 \leq t < 2$:



$$y(t) = \int_{-1}^1 e^{3(t-\tau)} [-e^{-2\tau}] d\tau$$

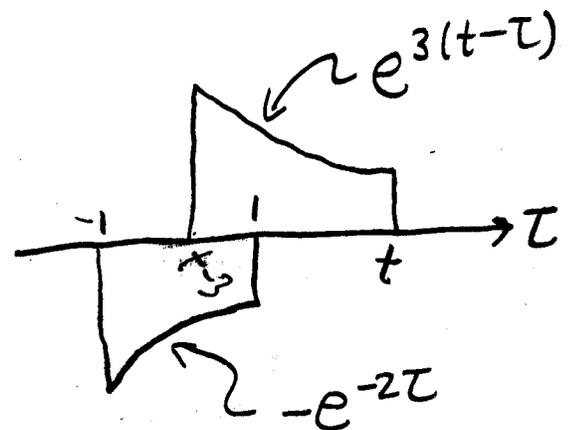
$$= -e^{3t} \int_{-1}^1 e^{-3\tau} e^{-2\tau} d\tau = -e^{3t} \int_{-1}^1 e^{-5\tau} d\tau$$

$$= \frac{1}{5} e^{3t} [e^{-5\tau}]_{\tau=-1}^1 = \frac{1}{5} e^{3t} [e^{-5} - e^5]$$

$$= \frac{1}{5} e^{3t-5} - \frac{1}{5} e^{3t+5}$$

Case IV) $t-3 > -1$ and $t-3 < 1$
 $t > 2$ and $t < 4$

$2 \leq t < 4$:



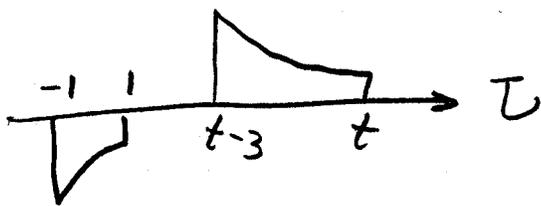
$$y(t) = \int_{t-3}^1 e^{3(t-\tau)} [-e^{-2\tau}] d\tau$$

$$= -e^{3t} \int_{t-3}^1 e^{-3\tau} e^{-2\tau} d\tau \quad \longrightarrow$$

Case IV) ...

$$\begin{aligned}
 &= -e^{3t} \int_{t-3}^1 e^{-5\tau} d\tau = \frac{1}{5} e^{3t} \left[e^{-5\tau} \right]_{\tau=t-3}^1 \\
 &= \frac{1}{5} e^{3t} \left[e^{-5} - e^{-5(t-3)} \right] \\
 &= \frac{1}{5} e^{3t} \left[e^{-5} - e^{-5t} e^{15} \right] \\
 &= \frac{1}{5} \left[e^{3t} e^{-5} - e^{3t} e^{-5t} e^{15} \right] \\
 &= \frac{1}{5} \left[e^{3t-5} - e^{-2t} e^{15} \right] = \frac{1}{5} e^{3t-5} - \frac{1}{5} e^{-2t+15}
 \end{aligned}$$

Case V) $t-3 \gg 1$: $t \gg 4$:



$$y(t) = 0.$$

All Together:

$$y(t) = \begin{cases} 0, & t < -1 \\ \frac{1}{5} e^{-2t} - \frac{1}{5} e^{3t+5}, & -1 \leq t < 1 \\ \frac{1}{5} e^{3t-5} - \frac{1}{5} e^{3t+5}, & 1 \leq t < 2 \\ \frac{1}{5} e^{3t-5} - \frac{1}{5} e^{-2t+15}, & 2 \leq t < 4 \\ 0, & t \geq 4 \end{cases}$$

Convolution With Deltas

- When you convolve a signal with a Dirac or a Kronecker delta, you can often write down the result without going through the usual steps of calculating the convolution integral or sum.

→ Convolution with Kronecker delta:

$$x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] = x[n] \cdot 1 = x[n]$$

one nonzero term, when $k=n$

→ Convolution with a shifted Kronecker delta just shifts the signal by the same amount:

$$x[n] * \delta[n-2] = \sum_{k=-\infty}^{\infty} x[n-k] \delta[k-2] = x[n-2]$$

one nonzero term, when $k=2$

NOTE: This implies that $x[n-3] * \delta[n-4] = x[n-7]$.

Also, $\delta[n-1] * \delta[n-2] = \delta[n-3]$.

NOTE: $\sum_{k=-\infty}^{10} x[k] \delta[k-5] = x[5],$

but $\sum_{k=-\infty}^2 x[k] \delta[k-5] = 0,$ because in the

second case the sum only goes from $k=-\infty$ to $k=2$, which doesn't include the place that the delta is "turned on".

Also, $\sum_{k=-\infty}^{10} x[n-k] \delta[k-5] = x[n-5],$

but $\sum_{k=-\infty}^2 x[n-k] \delta[k-5] = 0$ for the same reason as above.

NOTE: $x[n] * (\delta[n-1] + \delta[n-4]) = x[n] * \delta[n-1] + x[n] * \delta[n-4]$
 $= x[n-1] + x[n-4].$

→ Convolution with Dirac delta:

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau = x(t) \text{ by the "sifting property" of the Dirac delta.}$$

$$x(t) * \delta(t+5) = \int_{-\infty}^{\infty} x(t-\tau) \delta(\tau+5) d\tau$$

$$= x(t-\tau) \Big|_{\tau=-5} = x(t+5).$$

- So convolution with a shifted Dirac delta just shifts the signal by the same amount,

- As in discrete time,

$$\begin{aligned}x(t) * [\delta(t-1) - 5\delta(t+4)] \\&= x(t) * \delta(t-1) - 5x(t) * \delta(t+4) \\&= x(t-1) - 5x(t+4),\end{aligned}$$

$$\int_{-\infty}^{10} x(t) \delta(t-5) dt = x(5),$$

$$\text{but } \int_{-\infty}^2 x(t) \delta(t-5) dt = 0,$$

because in the second case the region of integration does not include the place where the delta is "turned on".

Likewise,

$$\int_{-\infty}^{10} x(t-\tau) \delta(\tau-5) d\tau = x(t-5),$$

$$\text{but } \int_{-\infty}^2 x(t-\tau) \delta(\tau-5) d\tau = \underline{\underline{0}}.$$

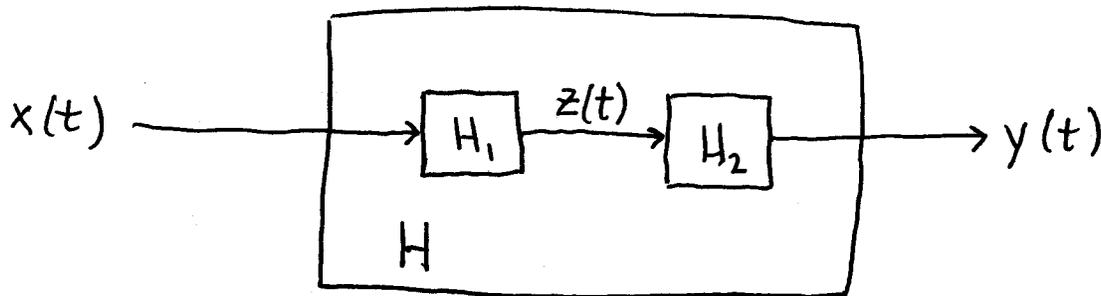
CASCADE CONNECTION OF TWO LSI SYSTEMS

- Suppose that H_1 and H_2 are two LSI systems with impulse responses $h_1(t)$ and $h_2(t)$:

$$x(t) \rightarrow \boxed{H_1} \rightarrow y_1(t) = x(t) * h_1(t)$$

$$x(t) \rightarrow \boxed{H_2} \rightarrow y_2(t) = x(t) * h_2(t)$$

- Consider a system H formed by connecting H_1 and H_2 in series:



$$\begin{aligned} y(t) &= H\{x(t)\} = H_2\{z(t)\} \\ &= H_2\{H_1\{x(t)\}\} \end{aligned}$$

- Question: is H linear?



- Let $x_1(t)$ and $x_2(t)$ be two arbitrary inputs and let a_1 and a_2 be constants.

$$\text{- Then } H \{ a_1 x_1(t) + a_2 x_2(t) \} = H_2 \left\{ H_1 \{ a_1 x_1(t) + a_2 x_2(t) \} \right\}$$

- Since H_1 is linear,

$$H_1 \{ a_1 x_1(t) + a_2 x_2(t) \} = a_1 H_1 \{ x_1(t) \} + a_2 H_1 \{ x_2(t) \}.$$

$$\text{- So } H \{ a_1 x_1(t) + a_2 x_2(t) \} = H_2 \left\{ a_1 H_1 \{ x_1(t) \} + a_2 H_1 \{ x_2(t) \} \right\}.$$

- But H_2 is also linear, so

$$\begin{aligned} & H_2 \left\{ a_1 H_1 \{ x_1(t) \} + a_2 H_1 \{ x_2(t) \} \right\} \\ &= a_1 H_2 \left\{ H_1 \{ x_1(t) \} \right\} + a_2 H_2 \left\{ H_1 \{ x_2(t) \} \right\} \\ &= a_1 H \{ x_1(t) \} + a_2 H \{ x_2(t) \}. \end{aligned}$$

- Therefore,

$$H \{ a_1 x_1(t) + a_2 x_2(t) \} = a_1 H \{ x_1(t) \} + a_2 H \{ x_2(t) \},$$

and the system H is linear.

- Question: is H shift invariant?

- Let $z(t) = H_1\{x(t)\}$.

- Then $y(t) = H_2\{z(t)\} = H_2\{H_1\{x(t)\}\}$.

- Lets input the shifted signal $x(t-t_0)$ to H:

$$\begin{aligned} H\{x(t-t_0)\} &= H_2\{H_1\{x(t-t_0)\}\} \\ &= H_2\{z(t-t_0)\} && \text{because } H_1 \text{ is shift invariant.} \\ &= y(t-t_0) && \text{because } H_2 \text{ is shift invariant.} \end{aligned}$$

- Therefore, H is shift invariant.

\Rightarrow H is an LSI system.

- Question: what is the impulse response of H?

$$z(t) = H_1\{x(t)\} = x(t) * h_1(t)$$

$$y(t) = H_2\{z(t)\} = z(t) * h_2(t)$$

$$= [x(t) * h_1(t)] * h_2(t)$$

$$= \int_{-\infty}^{\infty} [x(\tau) * h_1(\tau)] h_2(t-\tau) d\tau$$

\rightarrow

$$\dots y(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\theta) h_1(\tau - \theta) d\theta \right] h_2(t - \tau) d\tau$$

$$= \int_{-\infty}^{\infty} x(\theta) \left[\int_{-\infty}^{\infty} h_1(\tau - \theta) h_2(t - \tau) d\tau \right] d\theta$$

$$\begin{aligned} \psi &= \tau - \theta & \tau &= \psi + \theta \\ d\psi &= d\tau & d\tau &= d\psi \end{aligned}$$

$$= \int_{-\infty}^{\infty} x(\theta) \left[\int_{-\infty}^{\infty} h_1(\psi) h_2((t - \theta) - \psi) d\psi \right] d\theta$$

$$\underbrace{\hspace{10em}}_{h_1(t - \theta) * h_2(t - \theta)}$$

$$= \int_{-\infty}^{\infty} x(\theta) [h_1(t - \theta) * h_2(t - \theta)] d\theta$$

$$= x(t) * [h_1(t) * h_2(t)]$$

- Therefore, the impulse response of the system H is

$$h(t) = h_1(t) * h_2(t).$$

$$= \int_{-\infty}^{\infty} h_1(\tau) h_2(t - \tau) d\tau$$

$$= \int_{-\infty}^{\infty} h_2(\tau) h_1(t - \tau) d\tau$$

$$= h_2(t) * h_1(t).$$

NOTE: We have shown that, in general,

$$[x_1(t) * x_2(t)] * x_3(t) = x_1(t) * [x_2(t) * x_3(t)]$$

NOTE: We could also have found $h(t)$ for the cascade system H by putting in $\delta(t)$ and seeing what comes out:

$$\begin{aligned} h(t) &= H\{\delta(t)\} \\ &= H_2\{H_1\{\delta(t)\}\} \\ &= H_2\{h_1(t)\} \\ &= h_2(t) * h_1(t). \end{aligned}$$

SUMMARY:

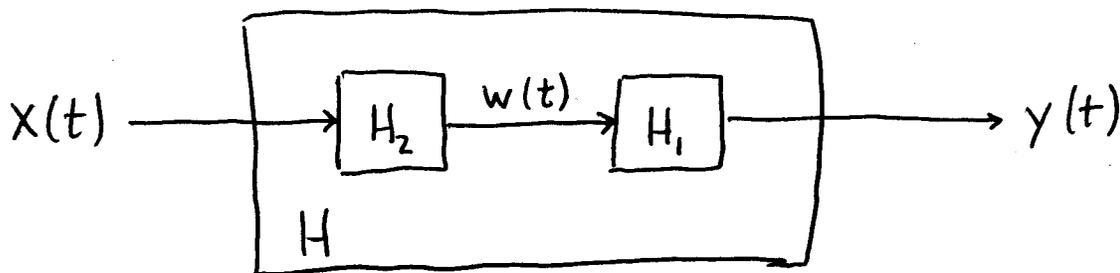
1. Convolution is associative:

$$(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$$

2. If H is a system formed by cascading two LSI systems with impulse responses h_1 and h_2 , then H is an LSI system with impulse response

$$h = h_1 * h_2.$$

- What if we switch the order of the connection:



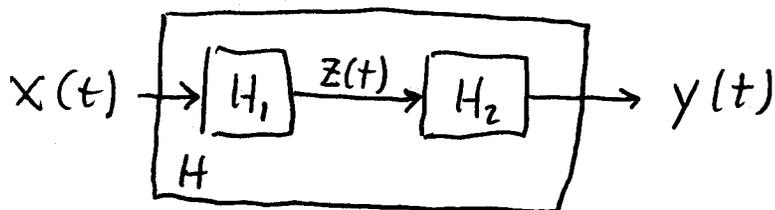
- Answer: The output $y(t)$ is the same.

→ The system above has impulse response

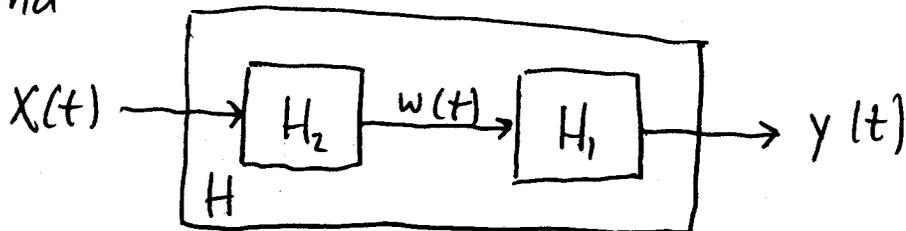
$$h(t) = h_2(t) * h_1(t)$$

$$= h_1(t) * h_2(t), \text{ since convolution is commutative.}$$

→ The systems



and



have the same impulse response.

→ For any input $x(t)$, both systems make the same output $y(t)$.

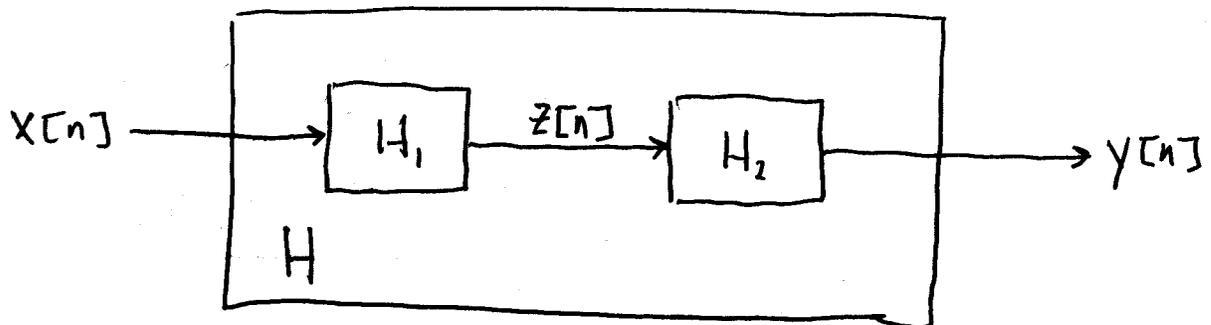
NOTE: Everything we have just done applies to discrete-time systems as well.

- Suppose H_1 and H_2 are two discrete-time LSI systems with unit pulse responses $h_1[n]$ and $h_2[n]$:

$$X[n] \rightarrow \boxed{H_1} \rightarrow Y_1[n] = X[n] * h_1[n]$$

$$X[n] \rightarrow \boxed{H_2} \rightarrow Y_2[n] = X[n] * h_2[n]$$

- Consider the system H formed by connecting H_1 and H_2 in series:



$$\text{- Then } y[n] = H_2 \{ z[n] \} = H_2 \{ H_1 \{ x[n] \} \} = H \{ x[n] \}.$$

- Since H_1 and H_2 are linear,

$$H \{ a_1 x_1[n] + a_2 x_2[n] \} = H_2 \{ H_1 \{ a_1 x_1[n] + a_2 x_2[n] \} \}$$

$$= H_2 \{ a_1 H_1 \{ x_1[n] \} + a_2 H_1 \{ x_2[n] \} \}$$

$$= a_1 H_2 \{ H_1 \{ x_1[n] \} \} + a_2 H_2 \{ H_1 \{ x_2[n] \} \}$$

$$= a_1 H \{ x_1[n] \} + a_2 H \{ x_2[n] \}$$

2.42

→ So H is linear.

- Furthermore, H_1 and H_2 are shift invariant, so

$$\begin{aligned} H\{x[n-n_0]\} &= H_2\{H_1\{x[n-n_0]\}\} \\ &= H_2\{z[n-n_0]\} \\ &= y[n-n_0], \end{aligned}$$

So H is shift invariant.

⇒ H is an LSI system.

- To find the impulse response of H , we will put in $\delta[n]$ and see what comes out.

→ we could also do this by putting in an arbitrary signal $x[n]$ and working out the convolutions (as we did for continuous-time)... Recommended exercise: do this.
★ ★.

→ If $x[n] = \delta[n]$,

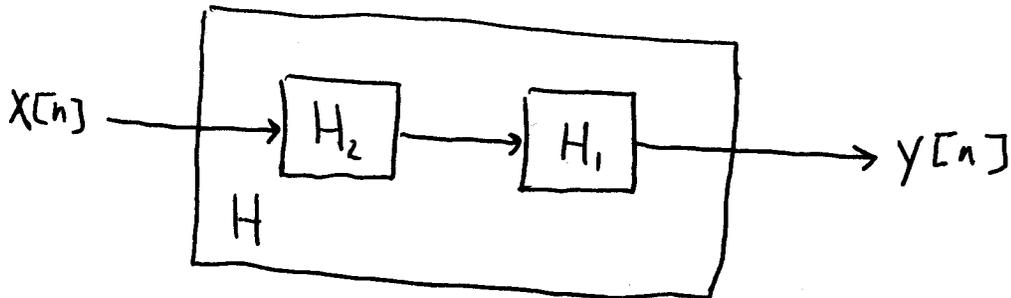
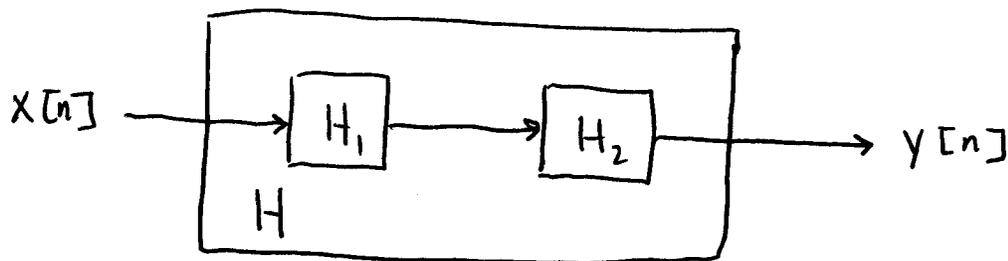
then $z[n] = \delta[n] * h_1[n] = h_1[n]$ comes out of H_1 .

So the unit pulse response of H is

$$h[n] = y[n] = z[n] * h_2[n] = h_1[n] * h_2[n]$$

- Since discrete-time convolution is commutative,
i.e., $h_1[n] * h_2[n] = h_2[n] * h_1[n]$,

it doesn't matter what order we connect the systems H_1 and H_2 up in:



- $y[n]$ is the same in either case.
- Both systems have the same impulse response $h[n] = h_1[n] * h_2[n] = h_2[n] * h_1[n]$.

PARALLEL CONNECTION OF TWO LSI SYSTEMS

- Suppose that H_1 and H_2 are two LSI systems with impulse responses h_1 and h_2 .

→ In continuous-time, the impulse responses are $h_1(t)$ and $h_2(t)$.

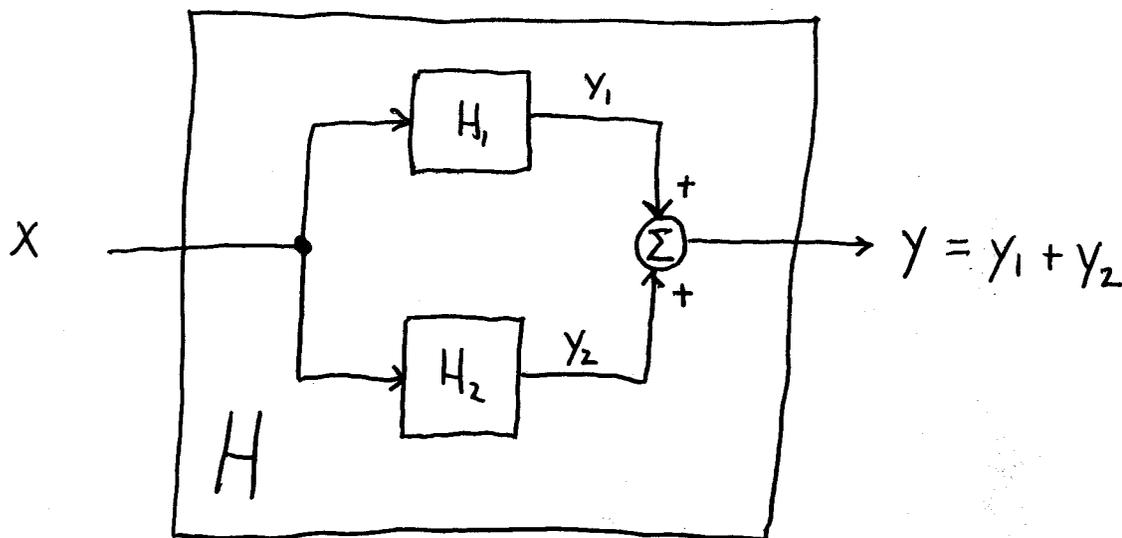
→ In discrete time, they are $h_1[n]$ and $h_2[n]$.

$$X \rightarrow \boxed{H_1} \rightarrow y = x * h_1$$

$$X \rightarrow \boxed{H_2} \rightarrow y = x * h_2$$

→ The picture is the same whether it is discrete time or continuous time.

- Consider a system H formed by connecting H_1 and H_2 in parallel:



- Discrete time: $y[n] = y_1[n] + y_2[n]$

- Continuous time: $y(t) = y_1(t) + y_2(t)$

- Question: Is H linear?

$$\begin{aligned} H\{a_1 x_1 + a_2 x_2\} &= H_1\{a_1 x_1 + a_2 x_2\} + H_2\{a_1 x_1 + a_2 x_2\} \\ &= a_1 H_1\{x_1\} + a_2 H_1\{x_2\} \quad \text{because } H_1 \text{ is linear.} \\ &\quad + a_1 H_2\{x_1\} + a_2 H_2\{x_2\} \quad \text{because } H_2 \text{ is linear} \\ &= a_1 [H_1\{x_1\} + H_2\{x_1\}] \\ &\quad + a_2 [H_1\{x_2\} + H_2\{x_2\}] \\ &= a_1 H\{x_1\} + a_2 H\{x_2\} \quad \checkmark \end{aligned}$$

→ The system H is linear.

- Question: is H shift invariant?

$$\begin{aligned} H\{x(t-t_0)\} &= H_1\{x(t-t_0)\} + H_2\{x(t-t_0)\} \quad (\text{continuous time case}) \\ &= y_1(t-t_0) + y_2(t-t_0) \quad \leftarrow \text{because } H_1 \text{ and } H_2 \text{ are shift invariant.} \\ &= y(t-t_0) \quad \checkmark \end{aligned}$$

$$\begin{aligned} H\{x[n-n_0]\} &= H_1\{x[n-n_0]\} + H_2\{x[n-n_0]\} \quad (\text{discrete time case}) \\ &= y_1[n-n_0] + y_2[n-n_0] \quad \leftarrow \text{because } H_1 \text{ and } H_2 \text{ are shift invariant} \\ &= y[n-n_0] \quad \checkmark \end{aligned}$$

→ The system H is shift invariant.

$\Rightarrow H$ is an LSI system.

- Question: what is the impulse response of H ?

\rightarrow Discrete time:

$$y[n] = y_1[n] + y_2[n]$$

$$= x[n] * h_1[n] + x[n] * h_2[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k] h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k] h_2[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k] (h_1[n-k] + h_2[n-k])$$

$$= x[n] * \underbrace{(h_1[n] + h_2[n])}_{h[n]}$$

$$\Rightarrow h[n] = h_1[n] + h_2[n]$$

- We could also have found this by putting in $\delta[n]$.

$\rightarrow H_1$ would then output $y_1[n] = h_1[n]$.

$\rightarrow H_2$ would then output $y_2[n] = h_2[n]$.

\rightarrow Output of H would be

$$h[n] = y[n] = h_1[n] + h_2[n].$$

→ Continuous Time:

$$\begin{aligned}y(t) &= y_1(t) + y_2(t) \\&= x(t) * h_1(t) + x(t) * h_2(t) \\&= \int_{-\infty}^{\infty} x(\tau) h_1(t-\tau) d\tau + \int_{-\infty}^{\infty} x(\tau) h_2(t-\tau) d\tau \\&= \int_{-\infty}^{\infty} x(\tau) [h_1(t-\tau) + h_2(t-\tau)] d\tau \\&= x(t) * \underbrace{(h_1(t) + h_2(t))}_{h(t)}\end{aligned}$$

$$\Rightarrow h(t) = h_1(t) + h_2(t).$$

NOTE: Since addition of real (and complex) numbers is commutative,

$$h[n] = h_1[n] + h_2[n] = h_2[n] + h_1[n]$$

$$h(t) = h_1(t) + h_2(t) = h_2(t) + h_1(t)$$

→ The "order" of the parallel connection does not matter.

SYSTEM PROPERTIES REVISITED

- We will now look again at several properties like causality and stability.
- We will restate these properties in terms of the unit pulse response $h[n]$ (discrete time) and impulse response $h(t)$ (continuous time).

NOTE: Two properties, linearity and shift invariance, will not be revisited.

→ Since we will be formulating system properties in terms of $h(t)$ and $h[n]$, it is implied that we are talking about systems that are both linear and shift invariant.

→ That is, we are concerned here with LSI systems.

Memoryless

- A discrete-time LSI system is memoryless if and only if

$$y[n] = Kx[n]$$

for some constant K .

- This implies that $h[n] = K\delta[n]$, so that

$$\begin{aligned}y[n] &= x[n] * h[n] \\&= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \\&= \sum_{k=-\infty}^{\infty} x[k] K\delta[n-k] \quad \leftarrow \text{one nonzero term,} \\&\quad \text{when } n=k. \\&= Kx[n]. \quad \checkmark\end{aligned}$$

- A continuous-time LSI system is memoryless if and only if

$$y(t) = Kx(t)$$

for some constant K .

- This implies that $h(t) = K\delta(t)$, so that

$$\begin{aligned}y(t) &= x(t) * h(t) \\&= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \\&= \int_{-\infty}^{\infty} x(\tau) K\delta(t-\tau) d\tau \quad \leftarrow \text{symbolic integral} \\&= \langle x(\tau), K\delta(t-\tau) \rangle \quad \leftarrow \text{assuming } K \in \mathbb{R}. \\&= K \langle x(\tau), \delta(t-\tau) \rangle \\&= Kx(t) \quad \checkmark\end{aligned}$$

Causality

FACT: a discrete-time LSI system H is causal if and only if $h[n] = 0 \quad \forall n < 0$.

Proof: Since the statement is "if and only if", we have two things to prove:

- 1) $h[n] = 0 \quad \forall n < 0 \rightarrow$ causal
- 2) causal $\rightarrow h[n] = 0 \quad \forall n < 0$.

Suppose $h[n] = 0 \quad \forall n < 0$. Then

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k] \\ &= \sum_{k=0}^{\infty} h[k] x[n-k]. \end{aligned}$$

So $y[n]$ depends on the current input $x[n]$ and on past inputs $x[n-k]$, $k > 0$, but not on future inputs $x[n-k]$, $k < 0$.

The system is therefore causal.

Now suppose the system is causal. When the input is $\delta[n]$, the output is $h[n]$. But $\delta[n] = 0 \quad \forall n < 0$, and, since the system is causal, this implies that the output $h[n]$ is zero $\forall n < 0$. Thus, $h[n] = 0 \quad \forall n < 0$.

QED.

FACT: A continuous-time LSI system is causal if and only if $h(t) = 0 \quad \forall t < 0$.

Proof: Suppose $h(t) = 0 \quad \forall t < 0$. Then

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\ &= \int_0^{\infty} h(\tau) x(t-\tau) d\tau. \end{aligned}$$

Thus, $y(t)$ depends on the current input $x(t)$ and on past inputs $x(t-\tau)$, $\tau > 0$, but not on future inputs $x(t-\tau)$ when $\tau < 0$. The system is therefore causal.

Now assume that the system is causal. When the input is $\delta(t)$, the output is $h(t)$. But $\delta(t) = 0 \quad \forall t < 0$. Since the system is causal, this implies $h(t) = 0 \quad \forall t < 0$.

QED



Stability

- FACT: a discrete-time LSI system H is stable if and only if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

(That is, iff $h[n] \in \ell^1(\mathbb{Z})$).

Proof:

Let $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ and let $x[n]$ be an arbitrary bounded input signal. Then $\exists B \in \mathbb{R}, B \geq 0$, such that $|x[n]| \leq B \forall n \in \mathbb{Z}$. Now,

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k].$$

So,

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k] x[n-k]|$$

$$= \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]| B$$

$$= B \sum_{k=-\infty}^{\infty} |h[k]| < \infty.$$

Thus, $y[n]$ is bounded and all bounded inputs produce bounded outputs. The system is therefore stable.

It remains to show that stable implies $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

We will construct a proof by contrapositive.

That is, we will prove that $A \rightarrow B$ by showing that $\bar{B} \rightarrow \bar{A}$ ("B not" implies "A not"). The two statements $A \rightarrow B$ and $\bar{B} \rightarrow \bar{A}$ are equivalent.

In this case, $A =$ "the system is stable" and $B = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

So, we will show that $\sum_{n=-\infty}^{\infty} |h[n]| = \infty$ implies that the system is not stable.

Suppose that $\sum_{n=-\infty}^{\infty} |h[n]| = \infty$.

$$\text{Let } x[n] = \begin{cases} \frac{h^*[-n]}{|h[-n]|} & , \quad h[-n] \neq 0 \\ 0 & , \quad h[-n] = 0 \end{cases}$$

Then $|x[n]| \leq 1$, so $x[n]$ is a bounded input.

→

$$\text{Let } a = \{k; h[-k] \neq 0\}.$$

Then

$$y[0] = \sum_{k=-\infty}^{\infty} x[k] h[0-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k] h[-k]$$

$$= \sum_{k \in a} x[k] h[-k] + \sum_{k \notin a} x[k] h[-k]$$

↑ ↑
both zero

$$= \sum_{k \in a} \frac{h^*[-k] h[-k]}{|h[-k]|} + \sum_{k \notin a} 0$$

$$= \sum_{k \in a} \frac{|h[-k]|^2}{|h[-k]|} + \sum_{k \notin a} |h[-k]|$$

↑ all terms are zero

$$= \sum_{k \in a} |h[-k]| + \sum_{k \notin a} |h[-k]|$$

$$= \sum_{m=-\infty}^{\infty} |h[m]| = \infty.$$

So the bounded input $x[n]$ produced an output $y[n]$ that is not bounded at $n=0$.

Therefore, the system is not stable.

QED.

2.55

FACT: a continuous-time LSI system H is stable if and only if $\int_{-\infty}^{\infty} |h(t)| dt < \infty$.

Proof: Suppose $\int_{-\infty}^{\infty} |h(t)| dt < \infty$, and let $x(t)$ be an arbitrary bounded input. Then $\exists B \in \mathbb{R}, B > 0$, such that $|x(t)| \leq B \quad \forall t \in \mathbb{R}$. Now,
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau.$$

$$\begin{aligned} \text{So, } |y(t)| &= \left| \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \right| \\ &\leq \int_{-\infty}^{\infty} |h(\tau) x(t-\tau)| d\tau \\ &= \int_{-\infty}^{\infty} |h(\tau)| |x(t-\tau)| d\tau \\ &\leq \int_{-\infty}^{\infty} |h(\tau)| B d\tau \\ &= B \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty. \end{aligned}$$

Thus, every bounded input produces a bounded output and the system is stable.

Now, suppose $\int_{-\infty}^{\infty} |h(t)| dt = \infty$.

$$\text{Let } x(t) = \begin{cases} \frac{h^*(-t)}{|h(-t)|}, & h(-t) \neq 0 \\ 0, & h(-t) = 0 \end{cases}$$

Then $|x(t)| \leq 1$ and $x(t)$ is a bounded input.

Let $a = \{t : h(-t) \neq 0\}$.

$$\begin{aligned} \text{Then } y(0) &= \int_{-\infty}^{\infty} x(\tau) h(0-\tau) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) h(-\tau) d\tau \\ &= \int_a \frac{h^*(-\tau) h(-\tau)}{|h(-\tau)|} d\tau + \int_{\mathbb{R} \setminus a} x(\tau) h(-\tau) d\tau \\ &= \int_a \frac{|h(-\tau)|^2}{|h(-\tau)|} d\tau + \int_{\mathbb{R} \setminus a} 0 d\tau \quad \begin{array}{l} \uparrow \uparrow \\ \text{both zero} \end{array} \\ &= \int_a |h(-\tau)| d\tau + \int_{\mathbb{R} \setminus a} |h(-\tau)| d\tau \quad \begin{array}{l} \uparrow \\ \text{zero} \end{array} \\ &= \int_{-\infty}^{\infty} |h(\tau)| d\tau = \infty. \end{aligned}$$

Since the bounded input $x(t)$ produced an output $y(t)$ that is not bounded at $t=0$, the system is not stable.

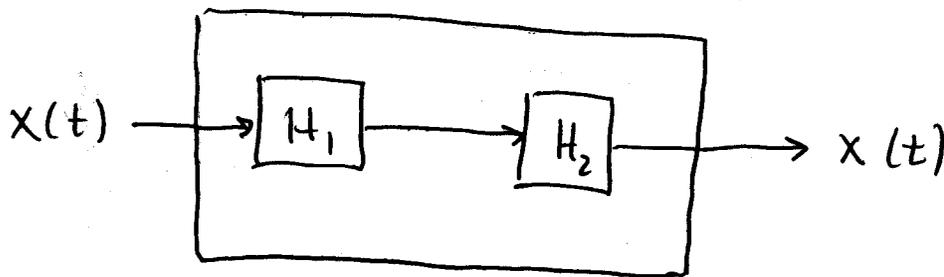
Thus, by proving the first half directly and the second half by contrapositive, we have shown that a continuous-time LSI system is stable if and only if $\int_{-\infty}^{\infty} |h(t)| dt < \infty$.

QED.

Invertibility

- The continuous-time LSI system with impulse response $h(t) = \delta(t)$ is called the identity system.
- The output of this system is equal to the input, since
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau = x(t).$$

↑ symbolic integral
- Suppose that H_1 is a continuous-time LSI system that is invertible.
- Let H_2 be the inverse system.
- This implies that the cascade connection of H_1 and H_2 is the identity system:



- Thus, in general, an LSI system H_1 is invertible if there exists an LSI system H_2 such that

$$h_1(t) * h_2(t) = \delta(t).$$

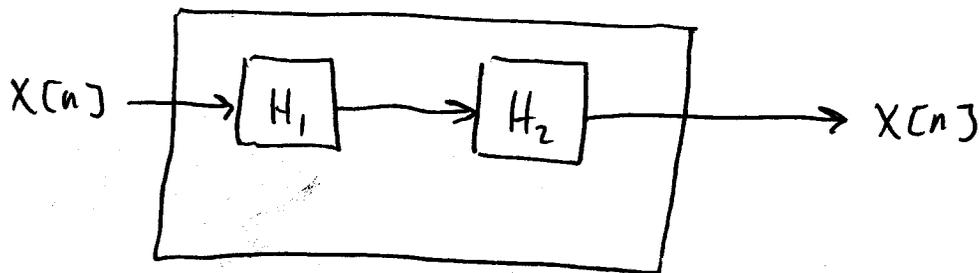
- Likewise, a discrete-time LSI system with unit pulse response $h[n] = \delta[n]$ is called the identity system.

- The output of this system is equal to the input, since

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] = x[n].$$

- Suppose that H_1 is a discrete time invertible LSI system and that H_2 is the inverse system.

- Then the cascade connection of H_1 and H_2 must be the identity system:



- Then the cascade connection must have unit pulse response $h[n] = h_1[n] * h_2[n] = \delta[n]$.

- This implies that a discrete time LSI system H_1 is invertible if there exists another LSI system H_2 such that

$$h_1[n] * h_2[n] = \delta[n].$$

Unit Step Response

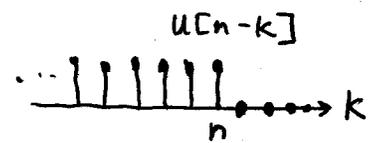
- We have been characterizing LSI systems in terms of their unit impulse responses.
- It is also possible to study LSI systems in terms of their response to the unit step. This response is called the "unit step response".

→ For a discrete-time LSI system, the unit step response $s[n]$ is the output when the input is $u[n]$.

→ For a continuous-time LSI system, the unit step response $s(t)$ is the output when the input is $u(t)$.

- Thus,

$$\begin{aligned} s[n] &= u[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] u[n-k] \\ &= \sum_{k=-\infty}^n h[k] \end{aligned}$$



$$\rightarrow \text{So, } s[n] - s[n-1] = \sum_{k=-\infty}^n h[k] - \sum_{k=-\infty}^{n-1} h[k]$$

$$= h[n] + \sum_{k=-\infty}^{n-1} (h[k] - h[k])$$

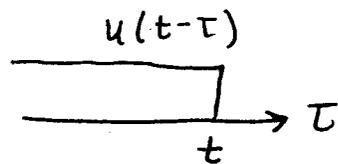
$$= h[n]$$

→ Then the relationship between $h[n]$ and $s[n]$ is

$$h[n] = s[n] - s[n-1].$$

- Likewise, for a continuous-time LSI system,

$$\begin{aligned} s(t) &= u(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau \\ &= \int_{-\infty}^t h(\tau) d\tau. \end{aligned}$$



- Applying the fundamental theorem of calculus, we see that the relationship between $s(t)$ and $h(t)$ is

$$h(t) = \frac{d}{dt} s(t).$$

~~We are done with Chapter 2!!!~~

Block Diagram Representation - 1st order Systems

- often, the systems we are interested in will have inputs and outputs that satisfy linear constant coefficient difference equations or differential equations:

$$x[n] \rightarrow \boxed{H} \rightarrow y[n]$$

- General Form of L.C.C. difference equation:

$$\sum_{k=0}^N \alpha_k y[n-k] = \sum_{m=0}^M \beta_m x[n-m]$$

- or

$$\alpha_0 y[n] + \alpha_1 y[n-1] + \dots + \alpha_N y[n-N] \\ = \beta_0 x[n] + \beta_1 x[n-1] + \dots + \beta_M x[n-M]$$

$$x(t) \rightarrow \boxed{H} \rightarrow y(t)$$

- General form of L.C.C. differential equation:

$$\sum_{k=0}^N \alpha_k \frac{d^k}{dt^k} y(t) = \sum_{m=0}^M \beta_m \frac{d^m}{dt^m} x(t)$$

- or

$$\alpha_0 y(t) + \alpha_1 y'(t) + \dots + \alpha_N y^{(N)}(t) = \beta_0 x(t) + \beta_1 x'(t) + \dots + \beta_M x^{(M)}(t)$$

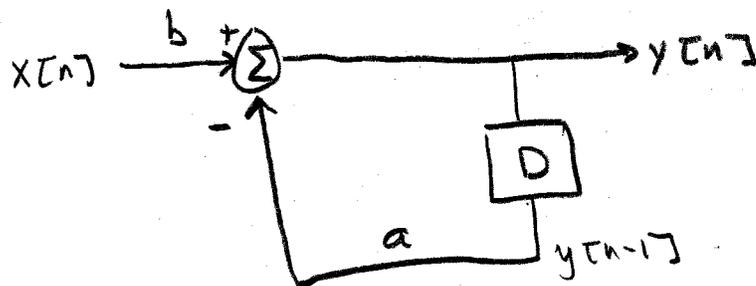
NOTE: These equations do not specify the system uniquely. For any such equation, there are many systems with inputs and outputs that satisfy the given equation.

- Simple Example of a difference equation:

$$y[n] + ay[n-1] = bx[n]$$

$$\Rightarrow y[n] = -ay[n-1] + bx[n]$$

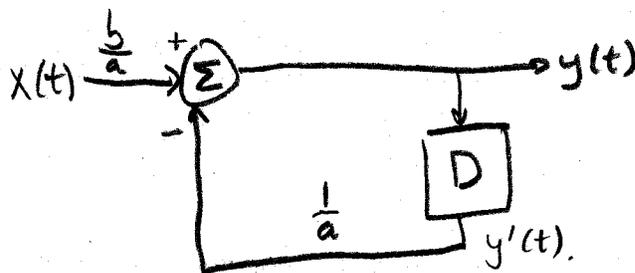
Block Diagram:



- Simple Example of a differential equation:

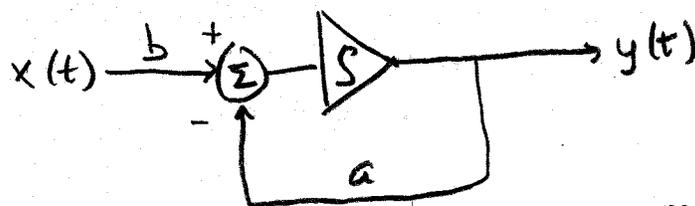
$$y'(t) + ay(t) = bx(t)$$

Block Diagram 1: $y(t) = -\frac{1}{a}y'(t) + \frac{b}{a}x(t)$



Block Diagram 2: $y'(t) = -ay(t) + bx(t)$

$$y(t) = \int_{-\infty}^t -ay(\tau) + bx(\tau) d\tau$$



We are done with Chapter 2!