

Chapter 4

Brief review of Eigenvectors & Eigenvalues

- If A is a matrix, \vec{u} is a vector, and $\lambda \in \mathbb{C}$ is a complex constant,

- and if $A\vec{u} = \lambda\vec{u}$,

\Rightarrow Then \vec{u} is called an "eigenvector" of the matrix A with associated eigenvalue λ .

EX: $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$A\vec{u} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = 4\vec{u}.$$

$\rightarrow \vec{u}$ is an eigenvector of the matrix A with associated eigenvalue $\lambda = 4$.

\rightarrow Multiplication by A doesn't change the direction of \vec{u} , it just "stretches" the length of the vector.

- For a system H , there may be certain inputs $x(t)$ $\{x[n]\}$ such that the output is a complex number times the input:

$$x(t) \rightarrow \boxed{H} \rightarrow y(t) = \lambda x(t)$$

$$x[n] \rightarrow \boxed{H} \rightarrow y[n] = \lambda x[n]$$

→ In these cases, we call $x(t)$ $\{x[n]\}$ an eigenfunction of the system H with associated eigenvalue λ .

FACT: The signal $e^{j\omega_0 n}$, for any fixed $\omega_0 \in \mathbb{R}$, is an eigenfunction for any discrete-time LSI system.

FACT: The signal $e^{j\omega_0 t}$, for any fixed $\omega_0 \in \mathbb{R}$, is an eigenfunction of any continuous-time LSI system.

$$e^{j\omega_0 n} \rightarrow \boxed{\begin{array}{c} H \\ \text{(LSI)} \end{array}} \rightarrow \lambda e^{j\omega_0 n}$$

$$e^{j\omega_0 t} \rightarrow \boxed{\begin{array}{c} H \\ \text{(LSI)} \end{array}} \rightarrow \lambda e^{j\omega_0 t}$$

⇒ The eigenvalue λ generally depends on the specific value of ω_0 ...

Given an LSI system H , each different choice of ω_0 will give you a distinct input signal with its own distinct eigenvalue.

Proof that $e^{j\omega_0 t}$ is an eigenfunction of any continuous-time LSI system:

Let H be a continuous-time LSI system with impulse response $h(t)$. Let the system input be $x(t) = e^{j\omega_0 t}$ for some fixed $\omega_0 \in \mathbb{R}$.

$$x(t) = e^{j\omega_0 t} \rightarrow \boxed{H} \rightarrow y(t) = x(t) * h(t)$$

Then the system output is given by

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0(t-\tau)} h(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega_0 \tau} h(\tau) d\tau$$

$$= e^{j\omega_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau$$

$$= x(t) \langle h(t), x(t) \rangle$$

$$= \underbrace{x(t)}_{\text{The input}} \underbrace{\lambda}_{\text{A complex number that depends on } \omega_0}$$

The input

A complex number that depends on ω_0 .

Proof that $e^{j\omega_0 n}$ is an eigenfunction of any discrete-time LSI system:

Let H be a discrete-time LSI system with impulse response $h[n]$. Let the system input be $x[n] = e^{j\omega_0 n}$ for some fixed $\omega_0 \in \mathbb{R}$.

$$x[n] = e^{j\omega_0 n} \rightarrow \boxed{H} \rightarrow y[n] = x[n] * h[n].$$

Then the system output is given by

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[n-k] h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega_0(n-k)} h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega_0 k} h[k] \\ &= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\ &= x[n] \langle h[n], x[n] \rangle \end{aligned}$$

$$= \underbrace{x[n]}_{\text{The input}} \underbrace{\lambda}_{\text{A complex number that depends on } \omega_0}$$

The input

A complex number that depends on ω_0 .

- In chapters 1 & 2, we wrote signals as linear combinations of shifted deltas.
 - Kronecker deltas in the discrete-time case
 - Dirac deltas in the continuous-time case.
- For an LSI system with a known impulse response, this made it easy to find the system output for an arbitrary input.
 - The output was given by convolution.
 - The "delta bases" were easy to think about because they corresponded to the "natural bases" $\{\vec{i}, \vec{j}\}$ in \mathbb{R}^2 and $\{\vec{i}, \vec{j}, \vec{k}\}$ in \mathbb{R}^3 .
 - But convolution is somewhat complicated to think about because it does not directly give us a closed form expression for the system output.
- In chapters 4 & 5, we will write signals as linear "compositions" of bases made from sinusoidal complex exponential signals.
 - Since each basis signal will then be an eigenfunction of any LSI system, this will make it easy to compute the system response to each term in the input "linear composition".
 - This will usually give us easy closed form expressions for the system output in response to any arbitrary input signal (when the input is written in terms of eigenfunctions).

- We will write continuous-time signals $x(t)$ as linear combinations of the basis set $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$.

→ This will be done using dot products as before.

- Given an LSI system H , we will write the eigenvalues for all of the basis signals $e^{j\omega t}$ together as a function $H(\omega)$ of ω .

NOTE: The book writes $H(j\omega)$, but this is not standard.

- Similarly, for a discrete-time LSI system H , we will write signals in terms of the basis set $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi]}$

→ Recall that discrete-time exponentials $e^{j\omega n}$ are only unique for $\omega \in [-\pi, \pi]$. Outside of this interval, you just get the same signals over again.

- For any LSI system H , we will write the eigenvalues for all the basis signals $e^{j\omega n}$ together as a function $H(e^{j\omega})$ of ω .

Note: In this case, the book's notation $H(e^{j\omega})$ is standard.

- For any LSI system H , the collection of eigenvalues $H(\omega)$ or $H(e^{j\omega})$ is called the Frequency Response of the system.

- Tabulated as a function of ω , the frequency response gives the eigenvalues associated with all complex exponential sinusoidal inputs $e^{j\omega t}$ or $e^{j\omega n}$.
- The eigenvalues are generally complex.
 - So $H(\omega)$ is a complex number for each ω .
 - $H(e^{j\omega})$ is a complex number for each ω .
- Thus, the frequency response can be written in rectangular form:

$$H(\omega) = \text{Re}[H(\omega)] + j \text{Im}[H(\omega)]$$

$$H(e^{j\omega}) = \text{Re}[H(e^{j\omega})] + j \text{Im}[H(e^{j\omega})]$$

OR in polar form:

$$H(\omega) = |H(\omega)| e^{j \arg H(\omega)}$$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j \arg H(e^{j\omega})}$$

Question: If the system input is a complex exponential (eigenfunction), how is it affected when it passes through the system?

⇒ In other words, since we know the output is a complex number times the input, what does the output look like?

Answer:

Continuous-time case: $x(t) = e^{j\omega t} = \cos \omega t + j \sin \omega t$

$$\begin{aligned}y(t) &= H(\omega) x(t) \\&= H(\omega) e^{j\omega t} \\&= |H(\omega)| e^{j \arg H(\omega)} e^{j\omega t} \\&= |H(\omega)| e^{j(\omega t + \arg H(\omega))} \\&= |H(\omega)| \cos[\omega t + \arg H(\omega)] + j |H(\omega)| \sin[\omega t + \arg H(\omega)]\end{aligned}$$

\Rightarrow Real and imaginary parts of $x(t)$ each get scaled by $|H(\omega)|$ and shifted by $\arg H(\omega)$.

Discrete-time case: $x[n] = e^{j\omega n} = \cos \omega n + j \sin \omega n$

$$\begin{aligned}y[n] &= H(e^{j\omega}) x[n] \\&= H(e^{j\omega}) e^{j\omega n} \\&= |H(e^{j\omega})| e^{j \arg H(e^{j\omega})} e^{j\omega n} \\&= |H(e^{j\omega})| e^{j[\omega n + \arg H(e^{j\omega})]} \\&= |H(e^{j\omega})| \cos[\omega n + \arg H(e^{j\omega})] + j |H(e^{j\omega})| \sin[\omega n + \arg H(e^{j\omega})]\end{aligned}$$

\Rightarrow Again, the real and imaginary parts of the input each get scaled by $|H(e^{j\omega})|$ and shifted by $\arg H(e^{j\omega})$.

- So, if the input to an LSI system can be written as a sum of complex exponentials, and if the eigenvalues are known, then it will be easy to calculate what the system output is.

Riemann-Lebesgue Lemma

- The theory of distributions, also known as generalized functions, describes signals in terms of how they behave in dot products and in systems.
- This is not the same as "probability distributions", and the two should not be confused.
- We don't have time to cover distribution theory in ECE 3793.
- But we do need one of the important results and its most important consequence.
- The Riemann-Lebesgue lemma states that, in the sense of distributions,
$$\lim_{\omega \rightarrow \infty} e^{j\omega t} = 0.$$
 - \Rightarrow This implies that $\lim_{\omega \rightarrow \infty} \cos \omega t = \lim_{\omega \rightarrow \infty} \sin \omega t = 0.$
 - \Rightarrow It means that in dot products and systems, as $\omega \rightarrow \infty$, the signals $e^{j\omega t}$, $\cos \omega t$, and $\sin \omega t$ all behave just like the signal $x(t) = 0.$

- The most important consequence of the Riemann-Lebesgue lemma is the following:

FACT: $\lim_{A \rightarrow \infty} \frac{\sin At}{\pi t} = \delta(t).$

- We will now use these results to see that the basis $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$ is orthogonal, but not orthonormal.

- When we use the dot product to write signals in terms of the basis, we will have to "fix things up" to account for this.

- Recall: a basis is orthonormal if the dot product of any basis vector with itself is one and the dot product of any two different basis vectors is zero.

- For the basis $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$,

→ The dot product of any two different basis vectors is zero, so the basis is orthogonal

→ But the dot product of any basis vector with itself is 2π instead of one, so the basis is not orthonormal.

- Lets show this:

$$\langle e^{j\omega_1 t}, e^{j\omega_2 t} \rangle = \int_{-\infty}^{\infty} e^{j\omega_1 t} e^{-j\omega_2 t} dt$$

$$= \lim_{A \rightarrow \infty} \int_{-A}^A e^{j(\omega_1 - \omega_2)t} dt$$

$$= \lim_{A \rightarrow \infty} \frac{1}{j(\omega_1 - \omega_2)} \left[e^{j(\omega_1 - \omega_2)t} \right]_{-A}^A$$

$$= \lim_{A \rightarrow \infty} \frac{1}{j(\omega_1 - \omega_2)} \left[e^{j(\omega_1 - \omega_2)A} - e^{-j(\omega_1 - \omega_2)A} \right]$$

$$= \lim_{A \rightarrow \infty} \frac{2}{\omega_1 - \omega_2} \left[\frac{e^{j(\omega_1 - \omega_2)A} - e^{-j(\omega_1 - \omega_2)A}}{2j} \right]$$

$$= \lim_{A \rightarrow \infty} \frac{2 \sin[A(\omega_1 - \omega_2)]}{\omega_1 - \omega_2}$$

$$= 2\pi \lim_{A \rightarrow \infty} \frac{\sin[A(\omega_1 - \omega_2)]}{\pi(\omega_1 - \omega_2)}$$

$$= 2\pi \delta(\omega_1 - \omega_2)$$

$$= \begin{cases} 0, & \omega_1 \neq \omega_2 \\ 2\pi \delta(0), & \omega_1 = \omega_2 \end{cases}$$

This is the norm squared

$$\text{So Length} = \sqrt{2\pi}$$

Recall: To write a signal in terms of an orthonormal basis, there are two steps:

① Take the dot product of the signal with each basis vector to compute coefficients.

② Add up the coefficients times their respective basis vectors to get your signal.

→ With the basis $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$, all of the coefficients will be too large by a factor of $\sqrt{2\pi}$, since the basis is not orthonormal.

→ In step ②, we have to divide by $\underline{2\pi}$ to correct for this, since each basis vector is also too long by $\sqrt{2\pi}$.

- Now, let's write an arbitrary continuous-time signal $x(t)$ in terms of the basis $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$.

→ For convenience, we will write the required coefficients from step ① all together as a function $X(\omega)$ of ω .

Step ①: compute the coefficients (dot products):

$$X(\omega) = \langle x(t), e^{j\omega t} \rangle = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad (*)$$

Step ②: Add up the coefficients times their respective basis signals to get $x(t)$. Remember to divide by 2π since the basis is not orthonormal:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad (**)$$

- Equation (*) is called the Fourier Transform of $x(t)$:

$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

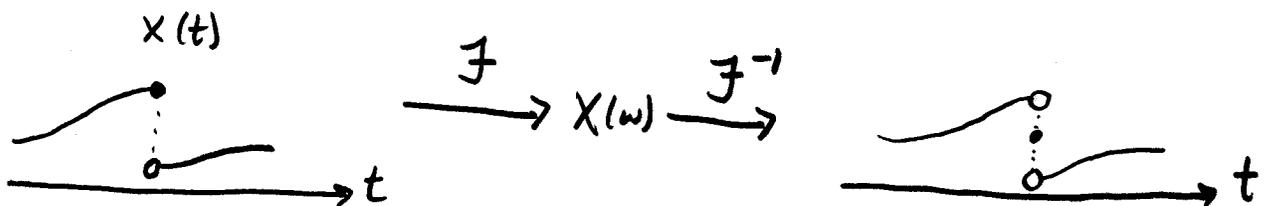
- Equation (***) is called the Inverse Fourier Transform of $X(\omega)$:

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

→ The coordinates $X(\omega)$ of the signal $x(t)$ with respect to the basis $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$ are called the frequency domain representation of $x(t)$.

NOTE: At points where $x(t)$ is discontinuous, $\mathcal{F}^{-1}[X(\omega)]$ converges to the midpoint of the discontinuity.

EX:



NOTE: There are four common ways that people deal with the fact that the basis $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$ is not orthonormal.

- This results in four different definitions for the Fourier transform.
- When you read other books, always be careful to note which definition is being used.
- The way our book does it is pretty much standard in electrical & computer engineering:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

- The other three ways are:

A) Fix up on the forward transform:

$$X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

B) Fix up equally on forward and reverse transforms:

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

C) Use the basis $\{e^{j2\pi ft}\}_{f \in \mathbb{R}}$, where "f" gives frequency in Hz instead of radians. This basis is orthonormal:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- But for ECE 3793, we will always use the Fourier transform definition that is in the book:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad (\text{forward transform})$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad (\text{reverse transform})$$

- We will often write

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

to indicate that $x(t)$ and $X(\omega)$ are a Fourier transform pair.

Remember: $\mathcal{F}^{-1}[X(\omega)]$ converges to the midpoints of discontinuities in $x(t)$.

Remember; The book writes $X(j\omega)$ instead of $X(\omega)$.

→ This is not standard.

→ I will write $X(\omega)$.

→ You can write $X(\omega)$ or $X(j\omega)$.

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- Generally, $X(\omega)$ is a complex number for each ω .

- Thus, we can write $X(\omega)$ in rectangular form:

$$X(\omega) = \text{Re}[X(\omega)] + j \text{Im}[X(\omega)]$$

- or in polar form:

$$X(\omega) = A(\omega) e^{j\phi(\omega)}$$

→ $A(\omega) = |X(\omega)|$ is called the "magnitude spectrum" of $x(t)$.

→ $\phi(\omega)$ is called the "spectral phase" of $x(t)$.

- Relationship between rectangular and polar forms:

$$A(\omega) = \sqrt{\{\text{Re}[X(\omega)]\}^2 + \{\text{Im}[X(\omega)]\}^2}$$

$$\phi(\omega) = \arctan \frac{\text{Im}[X(\omega)]}{\text{Re}[X(\omega)]}$$

$$\text{Re}[X(\omega)] = A(\omega) \cos[\phi(\omega)]$$

$$\text{Im}[X(\omega)] = A(\omega) \sin[\phi(\omega)]$$

EX: $x(t) = e^{-at} u(t)$, $a > 0$.

$x(t)$

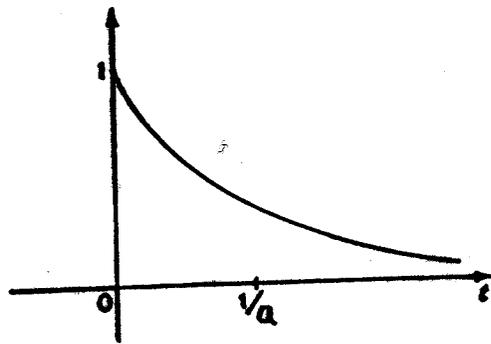
$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_{t=0}^{\infty}$$

$$= \frac{-1}{a+j\omega} \left[\lim_{A \rightarrow \infty} e^{-(a+j\omega)A} - 1 \right] = \frac{1}{a+j\omega}$$



- To get this in rectangular form, we conjugate the denominator:

$$X(\omega) = \frac{1}{a+j\omega} \frac{a-j\omega}{a-j\omega} = \frac{a}{a^2+\omega^2} - j \frac{\omega}{a^2+\omega^2}$$

$$\rightarrow \text{Re}[X(\omega)] = \frac{a}{a^2+\omega^2}$$

$$\rightarrow \text{Im}[X(\omega)] = \frac{-\omega}{a^2+\omega^2}$$

- For polar form,

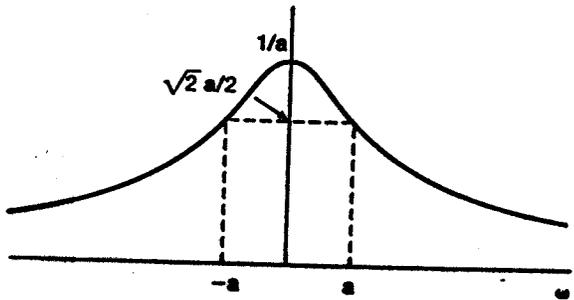
$$A(\omega) = |X(\omega)| = \sqrt{\frac{a^2}{(a^2+\omega^2)^2} + \frac{\omega^2}{(a^2+\omega^2)^2}}$$

$$= \sqrt{\frac{a^2+\omega^2}{(a^2+\omega^2)^2}} = \sqrt{\frac{1}{a^2+\omega^2}} = \frac{1}{\sqrt{a^2+\omega^2}}$$

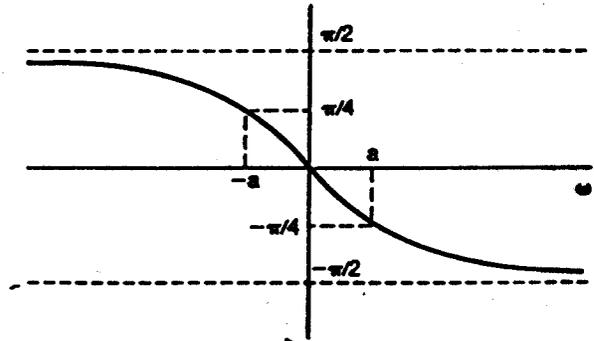
$$\phi(\omega) = \arg X(\omega) = \arctan \left[\frac{\frac{-\omega}{a^2 + \omega^2}}{\frac{a}{a^2 + \omega^2}} \right]$$

$$= \arctan \left(-\frac{\omega}{a} \right)$$

$A(\omega)$



$\phi(\omega)$



EX $x(t) = e^{-a|t|}$, $a > 0$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^0 e^{at} e^{-j\omega t} dt + \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_{-\infty}^0 e^{(a-j\omega)t} dt + \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{1}{a-j\omega} \frac{a+j\omega}{a+j\omega} + \frac{1}{a+j\omega} \frac{a-j\omega}{a-j\omega}$$

$$= \frac{a+j\omega}{a^2+\omega^2} + \frac{a-j\omega}{a^2+\omega^2} = \frac{2a}{a^2+\omega^2}$$

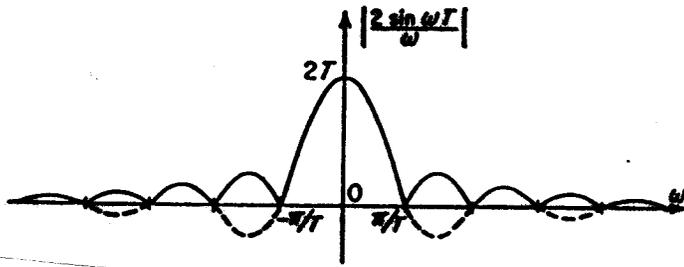
$$\operatorname{Re}[X(\omega)] = \frac{2 \sin \omega T}{\omega}$$

$$\operatorname{Im}[X(\omega)] = 0$$

$$A(\omega) = |X(\omega)| = 2 \left| \frac{\sin \omega T}{\omega} \right|$$

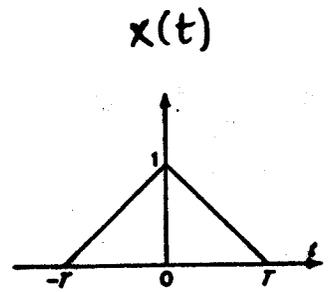
$$\phi(\omega) = \arg H(\omega) = 0$$

$X(\omega)$ ---
 $|X(\omega)|$ —



EX:

$$x(t) = \begin{cases} 1 - \frac{|t|}{T} & , |t| < T \\ 0 & , |t| \geq T \end{cases}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-T}^0 \left(1 + \frac{t}{T}\right) e^{-j\omega t} dt + \int_0^T \left(1 - \frac{t}{T}\right) e^{-j\omega t} dt$$

$$= \int_{-T}^0 e^{-j\omega t} dt + \frac{1}{T} \int_{-T}^0 t e^{-j\omega t} dt + \int_0^T e^{-j\omega t} dt - \frac{1}{T} \int_0^T t e^{-j\omega t} dt$$

$$= \int_{-T}^T e^{-j\omega t} dt + \frac{1}{T} \int_{-T}^0 t e^{-j\omega t} dt - \frac{1}{T} \int_0^T t e^{-j\omega t} dt$$

→

$$\dots X(\omega) = \int_{-T}^T \cos(-\omega t) dt + j \int_{-T}^T \sin(-\omega t) dt$$

ZERO... sine is odd

$$+ \frac{1}{T} \left[\frac{e^{-j\omega t}}{-j\omega} \left(t - \frac{1}{j\omega} \right) \right]_{t=-T}^0 - \frac{1}{T} \left[\frac{e^{-j\omega t}}{-j\omega} \left(t - \frac{1}{j\omega} \right) \right]_{t=0}^T$$

$$= \int_{-T}^T \cos \omega t dt + \frac{1}{T} \left[\frac{j e^{-j\omega t}}{\omega} \left(t - \frac{j}{\omega} \right) \right]_{-T}^0 - \frac{1}{T} \left[\frac{j e^{-j\omega t}}{\omega} \left(t - \frac{j}{\omega} \right) \right]_0^T$$

$$= \left[\frac{\sin \omega t}{\omega} \right]_{-T}^T + \frac{1}{T} \left[\frac{j}{\omega} \left(-\frac{j}{\omega} \right) - \frac{j e^{j\omega T}}{\omega} \left(-T - \frac{j}{\omega} \right) \right]$$

$$- \frac{1}{T} \left[\frac{j e^{-j\omega T}}{\omega} \left(T - \frac{j}{\omega} \right) - \frac{j}{\omega} \left(-\frac{j}{\omega} \right) \right]$$

$$= \frac{\sin \omega T - \sin(-\omega T)}{\omega} + \frac{1}{T\omega^2} - \frac{j e^{j\omega T}}{\omega} \left(-1 - \frac{j}{\omega T} \right)$$

$$- \frac{j e^{-j\omega T}}{\omega} \left(1 - \frac{j}{\omega T} \right) + \frac{1}{T\omega^2}$$

$$= \frac{2 \sin \omega T}{\omega} + \frac{2}{T\omega^2} + \frac{j e^{j\omega T}}{\omega} - \frac{e^{j\omega T}}{T\omega^2} - \frac{j e^{-j\omega T}}{\omega} - \frac{e^{-j\omega T}}{T\omega^2}$$

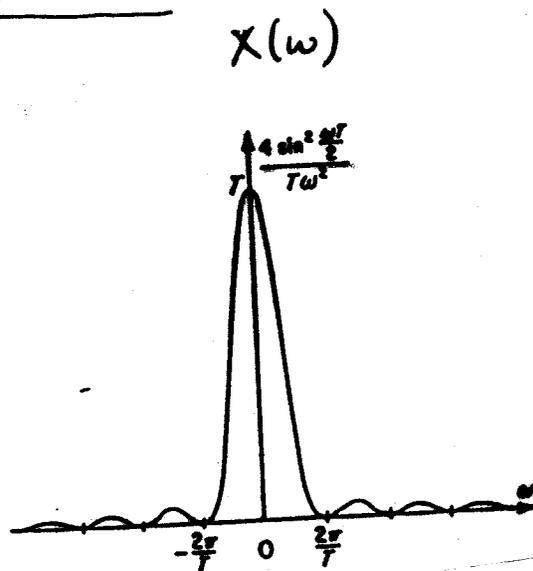
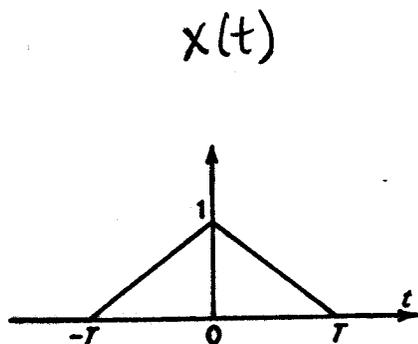
$$= \frac{2 \sin \omega T}{\omega} + \frac{2}{T\omega^2} - \frac{2}{\omega} \left[\frac{e^{j\omega T} - e^{-j\omega T}}{2j} \right] - \frac{2}{T\omega^2} \left[\frac{e^{j\omega T} - e^{-j\omega T}}{2} \right]$$

→

$$\dots X(\omega) = \underbrace{\frac{2 \sin \omega T}{\omega} + \frac{2}{T \omega^2}}_{\text{cancel}} - \underbrace{\frac{2}{\omega} \sin \omega T - \frac{2}{T \omega^2} \cos \omega T}_{\text{cancel}}$$

$$= \frac{2}{T \omega^2} [1 - \cos \omega T] = \frac{2}{T \omega^2} [2 \sin^2(\frac{\omega T}{2})]$$

$$X(\omega) = \frac{4 \sin^2(\frac{\omega T}{2})}{T \omega^2}$$



INVERSION

- Like the Fourier transform, the inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

generally requires integration.

- The forward and reverse (inverse) transform integrals are almost the same.

$$\rightarrow \text{F.T.} : e^{-j\omega t}$$

$$\text{I.F.T.} : e^{j\omega t}$$

$$\rightarrow \text{F.T.} : \int dt$$

$$\text{I.F.T.} : \frac{1}{2\pi} \int d\omega$$

- So the same types of integration techniques are used for computing both forward and inverse Fourier transforms.

- All of the preceding examples can be turned into examples of the inverse Fourier transform by:

1. Putting $\frac{1}{2\pi}$ out front.
2. Switching "t" and " ω " everywhere.
3. Changing $e^{-j\omega t}$ to $e^{j\omega t}$.

TABLES

- In practice, we don't always integrate to compute a Fourier transform or reverse transform.
- In fact, you should use integration only as a last resort.
- Where possible, try to find the transform you need in a table. Then you can just write down the answer.

FOURIER TRANSFORM PROPERTIES.

- We will now look at the main properties of the Fourier transform.
- Often, if you need a transform that is not in a table, you can use these properties to change the transform you need into one that is in the table. (or a combination of ones that are in the table).

Linearity :

$$\text{if } x_1(t) \xleftrightarrow{\mathcal{F}} X_1(\omega) \text{ and } x_2(t) \xleftrightarrow{\mathcal{F}} X_2(\omega)$$

and a_1 and a_2 are constants, then

$$a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{\mathcal{F}} a_1 X_1(\omega) + a_2 X_2(\omega).$$

Proof:
$$\begin{aligned} \mathcal{F}[a_1 x_1(t) + a_2 x_2(t)] &= \int_{-\infty}^{\infty} \{a_1 x_1(t) + a_2 x_2(t)\} e^{-j\omega t} dt \\ &= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt \\ &= a_1 X_1(\omega) + a_2 X_2(\omega). \quad \underline{\text{QED}}. \end{aligned}$$

Time Shifting:

$$\text{if } x(t) \xleftrightarrow{\mathcal{F}} X(\omega), \text{ then } x(t-t_0) \xleftrightarrow{\mathcal{F}} X(\omega) e^{-j\omega t_0}.$$

NOTE: $X(\omega) e^{-j\omega t_0} = A(\omega) e^{j[\phi(\omega) - t_0 \omega]}$

So time shifting introduces a linear phase shift in the spectrum.

Proof:
$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt \quad \begin{array}{l} u=t-t_0 \quad t=u+t_0 \\ du=dt \quad dt=du \end{array}$$

$$= \int_{-\infty}^{\infty} x(u) e^{-j\omega(u+t_0)} du$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(u) e^{-j\omega u} du = X(\omega) e^{-j\omega t_0}. \quad \underline{\text{QED}}.$$

Frequency Shifting:

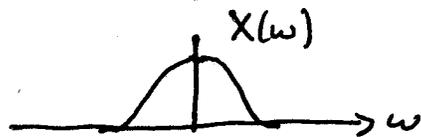
if $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$, then $e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0)$.

Proof:

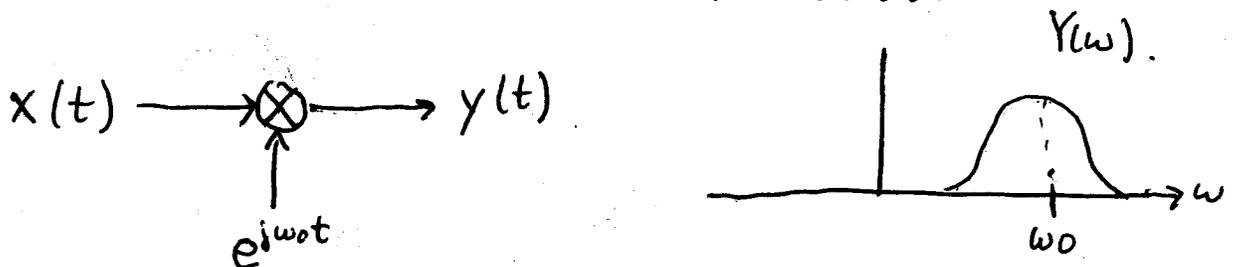
$$\begin{aligned} \mathcal{F}^{-1}[X(\omega - \omega_0)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega - \omega_0) e^{j\omega t} d\omega & \begin{array}{l} u = \omega - \omega_0 \quad \omega = u + \omega_0 \\ du = d\omega \quad d\omega = du \end{array} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(u) e^{j(u + \omega_0)t} du \\ &= e^{j\omega_0 t} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(u) e^{jut} du = e^{j\omega_0 t} x(t). \quad \underline{\text{QED.}} \end{aligned}$$

- The frequency shifting property is used often in radio and TV communications to shift a "baseband" program up to the carrier frequency of the station.

EX: $x(t)$ is a music signal with Fourier Transform



- If the station has an assigned frequency of ω_0 , here is what the transmitter can do:



→ This is called Amplitude Modulation, or AM.

Time Scaling:

$$\text{if } x(t) \xleftrightarrow{\mathcal{F}} X(\omega), \text{ then } x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right), a \neq 0.$$

Proof:

(i) Suppose $a > 0$. Then

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt \quad \begin{array}{l} u=at \quad t = \frac{u}{a} \\ du=adt \quad dt = \frac{du}{a} \end{array}$$

$$= \int_{-\infty}^{\infty} x(u) e^{-j\frac{\omega}{a}u} \frac{du}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(u) e^{-j\frac{\omega}{a}u} du = \frac{1}{a} X\left(\frac{\omega}{a}\right).$$

(ii) Suppose $a < 0$. Then

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt \quad \begin{array}{l} u=at \quad t = \frac{u}{a} \\ du=adt \quad dt = \frac{du}{a} \end{array}$$

$$= \int_{\infty}^{-\infty} x(u) e^{-j\omega \frac{u}{a}} \frac{du}{a} = -\frac{1}{a} \int_{-\infty}^{\infty} x(u) e^{-j\frac{\omega}{a}u} du$$

$$= \frac{1}{-a} X\left(\frac{\omega}{a}\right).$$

Combining parts (i) and (ii),

$$\mathcal{F}[x(at)] = \frac{1}{|a|} X\left(\frac{\omega}{a}\right), \quad \underline{\text{QED}}.$$

$a \neq 0$.

Frequency Scaling:

$$\text{if } x(t) \xleftrightarrow{\mathcal{F}} X(\omega), \text{ then } \frac{1}{|a|} x\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} X(a\omega), \quad a \neq 0.$$

Proof:

(i) Suppose $a > 0$. Then

$$\begin{aligned} \mathcal{F}^{-1}[X(a\omega)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(a\omega) e^{j\omega t} d\omega & u = a\omega & \quad \omega = \frac{u}{a} \\ & & du = a d\omega & \quad d\omega = \frac{du}{a} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(u) e^{j\frac{u}{a}t} \frac{du}{a} = \frac{1}{a} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(u) e^{ju\frac{t}{a}} du \\ &= \frac{1}{a} x\left(\frac{t}{a}\right). \end{aligned}$$

(ii) Suppose $a < 0$. Then

$$\begin{aligned} \mathcal{F}^{-1}[X(a\omega)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(a\omega) e^{j\omega t} d\omega & u = a\omega & \quad \omega = \frac{u}{a} \\ & & du = a d\omega & \quad d\omega = \frac{du}{a} \\ &= \frac{1}{2\pi} \int_{\infty}^{-\infty} X(u) e^{j\frac{u}{a}t} \frac{du}{a} = -\frac{1}{a} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(u) e^{ju\frac{t}{a}} du \\ &= -\frac{1}{a} x\left(\frac{t}{a}\right). \end{aligned}$$

Combining parts (i) and (ii),

$$\mathcal{F}^{-1}[X(a\omega)] = \frac{1}{|a|} x\left(\frac{t}{a}\right), \quad a \neq 0. \quad \underline{\text{Q.E.D.}}$$

NOTE: By combining the time scaling and frequency shifting properties, we get:

$$\text{if } x(t) \xleftrightarrow{\mathcal{F}} X(\omega), \text{ then } e^{j\omega_0 t} x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega - \omega_0}{a}\right).$$

Duality:

$$\text{if } x(t) \xleftrightarrow{\mathcal{F}} X(\omega), \text{ then } X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-\omega).$$

Proof: since $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$ by hypothesis, we have

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

$$\text{so } 2\pi x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad \begin{cases} \xi = -t \\ d\xi = -dt \end{cases} \quad \psi = \omega$$

$$\Rightarrow 2\pi x(-\xi) = \int_{-\infty}^{\infty} X(\psi) e^{-j\xi\psi} d\psi$$

now write ω instead of ξ and t instead of ψ :

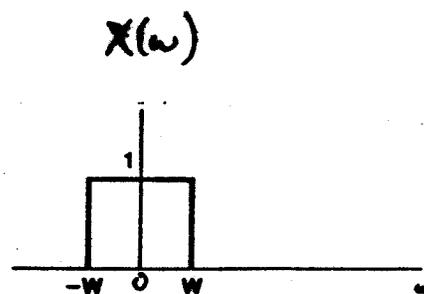
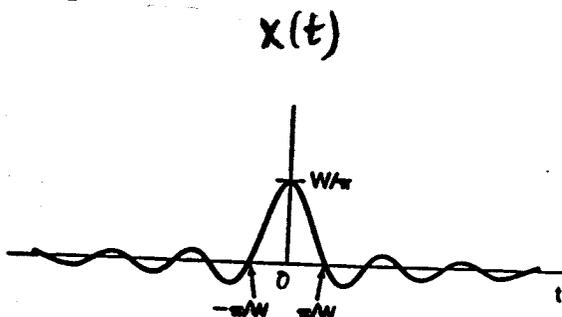
$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt = \mathcal{F}[X(t)].$$

QED.

EX: On page 4.27, we showed that

$$x(t) = \begin{cases} 1, & |t| \leq T \\ 0, & |t| > T \end{cases} \xleftrightarrow{\mathcal{F}} \frac{2 \sin \omega T}{\omega}$$

By duality, $\frac{\sin Wt}{\pi t} \xleftrightarrow{\mathcal{F}} X(\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$



Time Differentiation

if $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$, then $\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{F}} j\omega X(\omega)$.

Proof; By hypothesis, $x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

So $\frac{d}{dt}x(t) = \frac{d}{dt} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right\}$ (there's only one "t" in it)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left\{ \frac{d}{dt} e^{j\omega t} \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \{ j\omega e^{j\omega t} \} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ j\omega X(\omega) \} e^{j\omega t} d\omega$$

$$= \mathcal{F}^{-1}\{ j\omega X(\omega) \}. \quad \underline{\text{QED.}}$$

- Applying this property over and over n times, we get

$$\frac{d^n}{dt^n} x(t) \xleftrightarrow{\mathcal{F}} (j\omega)^n X(\omega)$$

Frequency Differentiation

if $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$, then $-jt x(t) \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega} X(\omega)$



Proof: by hypothesis, $\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = X(\omega)$.

$$\begin{aligned} \text{So } \frac{d}{d\omega} X(\omega) &= \frac{d}{d\omega} \left\{ \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right\} \\ &= \int_{-\infty}^{\infty} x(t) \left\{ \frac{d}{d\omega} e^{-j\omega t} \right\} dt \\ &= \int_{-\infty}^{\infty} x(t) (-jt) e^{-j\omega t} dt \\ &= \mathcal{F}[-jt x(t)]. \quad \underline{\text{QED.}} \end{aligned}$$

- Applying the frequency differentiation property n times repeatedly, we get

$$(-jt)^n x(t) \xleftrightarrow{\mathcal{F}} \frac{d^n}{d\omega^n} X(\omega)$$

Conjugation:

$$\text{if } x(t) \xleftrightarrow{\mathcal{F}} X(\omega), \text{ then } x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-\omega).$$

Proof: By hypothesis, $x(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$.

$$\text{So } X^*(\omega) = \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{\infty} x^*(t) e^{j\omega t} dt$$

→ write $-\xi$ for ω :

$$X^*(-\xi) = \int_{-\infty}^{\infty} x^*(t) e^{-j\xi t} dt$$

→ write ω for ξ :

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt = \mathcal{F}[x^*(t)]. \quad \underline{\text{QED.}}$$

NOTE: if $x(t)$ is REAL, then $x^*(t) = x(t)$.

- Applying the conjugation property, we then get

$$X^*(-\omega) = \mathcal{F}[x^*(t)] = \mathcal{F}[x(t)] = X(\omega).$$

- In other words,

\Rightarrow If $x(t)$ is real, then $X(\omega)$ is conjugate symmetric.

$$\rightarrow X(\omega) = X^*(-\omega)$$

\rightarrow The real part of $X(\omega)$ is even.

\rightarrow The imaginary part of $X(\omega)$ is odd.

\rightarrow This implies that $A(\omega) = |X(\omega)|$ is even:

$$\begin{aligned} |X(-\omega)| &= \sqrt{\{\operatorname{Re}[X(-\omega)]\}^2 + \{\operatorname{Im}[X(-\omega)]\}^2} \\ &= \sqrt{\{\operatorname{Re}[X(\omega)]\}^2 + \{-\operatorname{Im}[X(\omega)]\}^2} \\ &= \sqrt{\{\operatorname{Re}[X(\omega)]\}^2 + \{\operatorname{Im}[X(\omega)]\}^2} \\ &= |X(\omega)|, \quad \text{even} \end{aligned}$$

\rightarrow This also implies that $\phi(\omega) = \arg X(\omega)$ is odd:

$$\begin{aligned} \arg X(-\omega) &= \arctan \frac{\operatorname{Im}[X(-\omega)]}{\operatorname{Re}[X(-\omega)]} \\ &= \arctan \frac{-\operatorname{Im}[X(\omega)]}{\operatorname{Re}[X(\omega)]} \\ &= -\arctan \frac{\operatorname{Im}[X(\omega)]}{\operatorname{Re}[X(\omega)]} = -\arg X(\omega). \end{aligned}$$

odd

FACT: if $x(t)$ is a purely imaginary signal, then $X(\omega)$ is conjugate antisymmetric:

$$X(-\omega) = -X^*(\omega)$$

→ $\text{Re}[X(\omega)]$ is odd.

→ $\text{Im}[X(\omega)]$ is even.

→ $A(\omega) = |X(\omega)|$ is odd.

→ $\phi(\omega) = \arg X(\omega)$ is even.

⇒ Recommended Exercise: Show This. ★★

Even Signals:

if $x(t)$ is real-valued and even, then $X(\omega)$ is also real and even.

Proof: Let $x(t)$ be real and even. Then

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(t) \cos(-\omega t) dt + j \int_{-\infty}^{\infty} x(t) \sin(-\omega t) dt \\ &= \int_{-\infty}^{\infty} x(t) \cos \omega t dt - j \int_{-\infty}^{\infty} x(t) \sin \omega t dt \end{aligned}$$

But $x(t)$ is even, so $x(t) \sin \omega t$ is odd, and the second integral is zero:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cos \omega t dt, \text{ which is } \underline{\text{real}}.$$

→

Furthermore,

$$X(-\omega) = \int_{-\infty}^{\infty} x(t) \cos(-\omega t) dt = \int_{-\infty}^{\infty} x(t) \cos \omega t dt = X(\omega),$$

which establishes that $X(-\omega) = X(\omega)$. QED.

So: real and even \xleftrightarrow{F} real and even.

Odd Signals:

- In a similar manner, one can show that if $x(t)$ is real and odd, then $X(\omega)$ is pure imaginary and odd:

real and odd \xleftrightarrow{F} imaginary and odd.

Recommended Exercise: Show This. ★★

NOTE:

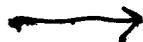
- Recall that any signal $x(t)$ can be uniquely written as the sum of an even signal and an odd signal:

$$x(t) = \mathcal{E}\{x(t)\} + \mathcal{O}\{x(t)\},$$

where

$$\mathcal{E}\{x(t)\} = \frac{1}{2} [x(t) + x(-t)]$$

$$\mathcal{O}\{x(t)\} = \frac{1}{2} [x(t) - x(-t)]$$



- Taking the Fourier transform of $x(t)$, we have

$$\begin{aligned}
 \mathcal{F}[x(t)] &= \mathcal{F}[\mathcal{E}\{x(t)\} + \mathcal{O}\{x(t)\}] \\
 &\overset{\text{pure real}}{=} \mathcal{F}[\mathcal{E}\{x(t)\}] + \mathcal{F}[\mathcal{O}\{x(t)\}] \quad \leftarrow \text{pure imaginary} \\
 &= \text{Re}[X(\omega)] + j \text{Im}[X(\omega)]
 \end{aligned}$$

\Rightarrow So,

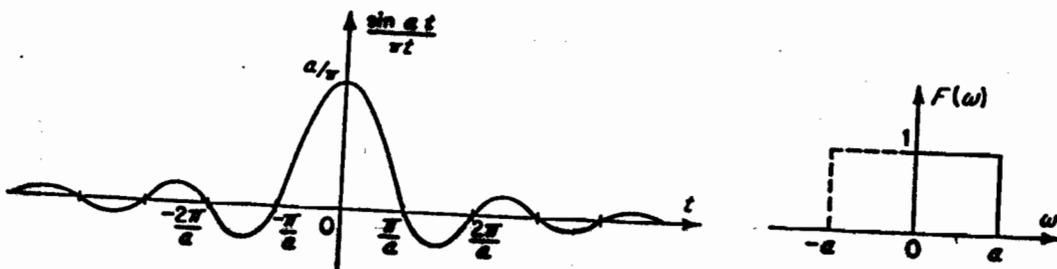
$\rightarrow \text{Re}[X(\omega)]$ is the Fourier transform of $\mathcal{E}\{x(t)\}$.

$\rightarrow j \text{Im}[X(\omega)]$ is the Fourier transform of $\mathcal{O}\{x(t)\}$.

- Now for some examples of the properties we have seen so far:

EX: Applying the duality property to the transform pair we derived on pages 4.26 and 4.27, we get

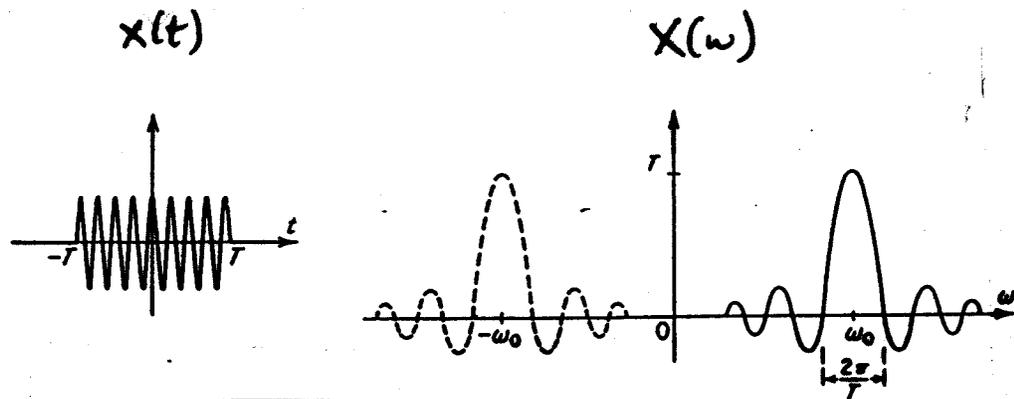
$$\frac{\sin at}{\pi t} \xleftrightarrow{\mathcal{F}} X(\omega) = \begin{cases} 1, & |\omega| < a \\ 0, & |\omega| > a \end{cases}$$



EX: $x(t) = \begin{cases} \cos \omega_0 t, & |t| \leq T \\ 0, & |t| > T \end{cases}$

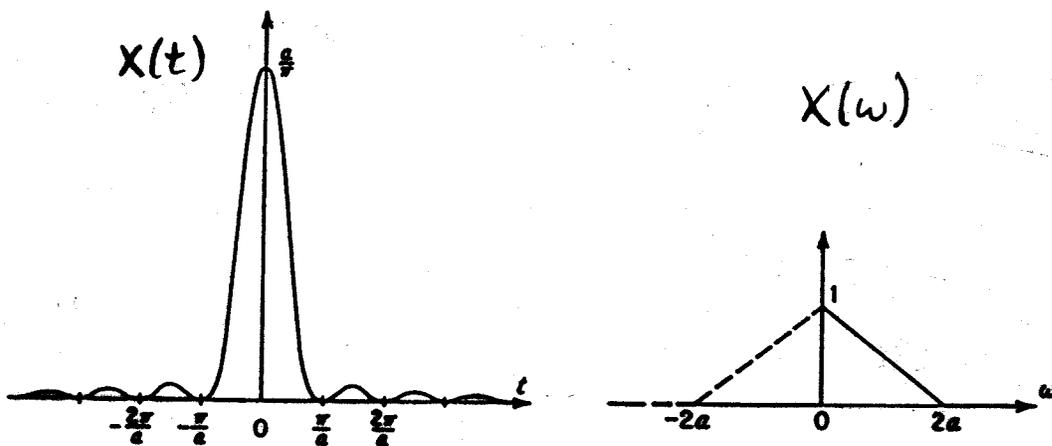
Applying the frequency shifting property to the transform pair we derived on pages 4.26 and 4.27 gives us

$$X(\omega) = \frac{\sin(\omega - \omega_0)T}{\omega - \omega_0} + \frac{\sin(\omega + \omega_0)T}{\omega + \omega_0}$$



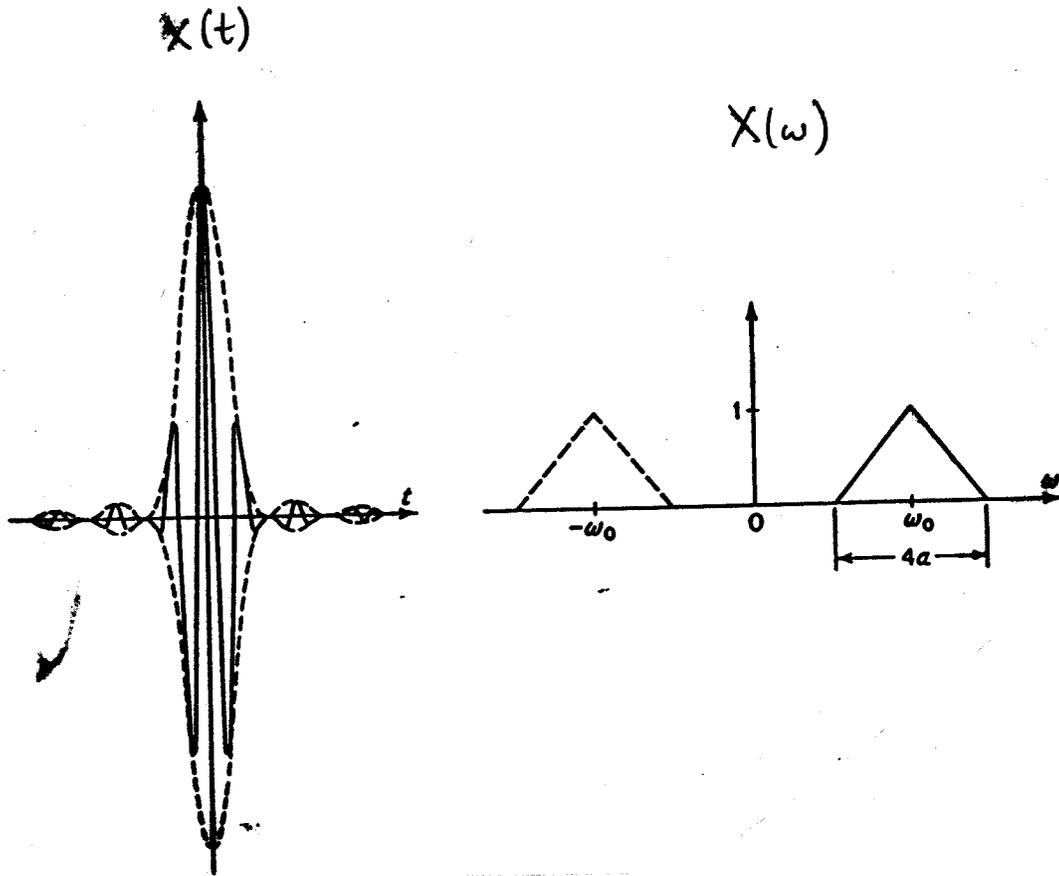
EX: Applying the duality property to the transform pair we derived at the bottom of page 4.29, we get

$$\frac{\sin^2 at}{\pi a t^2} \xleftrightarrow{\mathcal{F}} X(\omega) = \begin{cases} 1 - \frac{|\omega|}{2a}, & |\omega| < 2a \\ 0, & |\omega| > 2a \end{cases}$$



EX: Applying the frequency shifting property to the preceding example, we get

$$\frac{2 \sin^2 at}{\pi a t^2} \cos \omega_0 t = \frac{\sin^2 at}{\pi a t^2} \left(\frac{1}{2} [e^{j\omega_0 t} + e^{-j\omega_0 t}] \right) \xleftrightarrow{\mathcal{F}} X(\omega)$$



Time Convolution:

$$\text{if } x_1(t) \xleftrightarrow{\mathcal{F}} X_1(\omega) \text{ and } x_2(t) \xleftrightarrow{\mathcal{F}} X_2(\omega),$$

$$\text{then } x_1(t) * x_2(t) \xleftrightarrow{\mathcal{F}} X_1(\omega) X_2(\omega).$$



Proof:

$$\mathcal{F}[x_1(t) * x_2(t)] = \mathcal{F}\left[\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau\right]$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right] e^{-j\omega t} dt$$

Change
Integration
order

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau) e^{-j\omega t} dt \right] d\tau$$

Apply time
shifting
property

$$= \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega\tau} X_2(\omega) d\tau$$

$$= X_2(\omega) \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega\tau} d\tau$$

$$= X_2(\omega) X_1(\omega) = X_1(\omega) X_2(\omega). \quad \underline{\text{QED}}$$

NOTE: This is one of the most important properties we will learn this semester.

→ For an LSI system H ,

$$x(t) \rightarrow \boxed{H} \rightarrow y(t) = x(t) * h(t),$$

it says that

$$Y(\omega) = X(\omega) H(\omega),$$

$$\text{where } Y(\omega) = \mathcal{F}[y(t)]$$

$$\text{and } H(\omega) = \mathcal{F}[h(t)].$$

Frequency Convolution:

$$\text{if } x_1(t) \xleftrightarrow{\mathcal{F}} X_1(\omega) \text{ and } x_2(t) \xleftrightarrow{\mathcal{F}} X_2(\omega),$$

$$\text{then } x_1(t)x_2(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} [X_1(\omega) * X_2(\omega)].$$

Proof:

$$\begin{aligned} x_1(t)x_2(t) &= \mathcal{F}^{-1}[X_1(\omega)] \mathcal{F}^{-1}[X_2(\omega)] \\ &= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\alpha) e^{j\alpha t} d\alpha \right] \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\beta) e^{j\beta t} d\beta \right] \end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} X_1(\alpha) X_2(\beta) e^{j(\alpha+\beta)t} d\alpha \right] d\beta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} X_2(\beta) X_1(\alpha) e^{j(\alpha+\beta)t} d\alpha \right] d\beta$$

$$\omega = \alpha + \beta \quad \alpha = \omega - \beta$$

$$d\omega = d\alpha \quad d\alpha = d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} X_2(\beta) X_1(\omega - \beta) e^{j\omega t} d\omega \right] d\beta$$

Change order
of integration

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} X_2(\beta) X_1(\omega - \beta) d\beta \right] e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} [X_1(\omega) * X_2(\omega)] e^{j\omega t} d\omega$$

$$\Rightarrow = \mathcal{F}^{-1} \left\{ \frac{1}{2\pi} [X_1(\omega) * X_2(\omega)] \right\} \quad \underline{\text{QED.}}$$

Parseval's Formula:

if $x(t) \xleftrightarrow{F} X(\omega)$ and $X(\omega)$ is square-integrable,

$$\text{then } \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

Proof:

$$\text{By hypothesis, } x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

$$\text{So } x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega.$$

$$\text{Now, } \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) x^*(t) dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \right] dt$$

Change order
of integration

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) X(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega. \quad \underline{\text{QED.}}$$

DISTRIBUTIONAL FOURIER TRANSFORMS

- A general signal $x(t)$ can be written as

$$x(t) = x_a(t) + x_p(t),$$

where

→ $x_a(t)$ is a periodic

→ $x_p(t)$ is periodic.

- If $x_p(t)$ is not zero, then the Fourier transform

$X(\omega) = \mathcal{F}[x(t)]$ must contain Dirac deltas.

⇒ In other words, if $x(t)$ contains a periodic component, then $X(\omega)$ will contain Dirac deltas.

- Before looking at periodic signals, let's derive the Fourier transform of $\delta(t)$.

Forward Transform

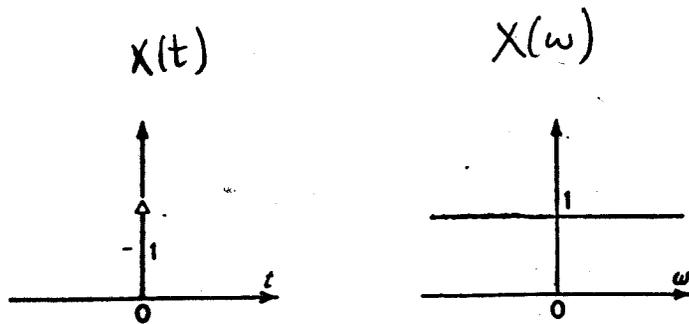
$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \langle e^{-j\omega t}, \delta(t) \rangle = 1.$$

Reverse Transform

$$\begin{aligned} \mathcal{F}^{-1}[1] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos \omega t d\omega + \underbrace{\frac{j}{2\pi} \int_{-\infty}^{\infty} \sin \omega t d\omega}_{\text{Zero (odd integrand)}} \\ &= \frac{1}{2\pi} \lim_{A \rightarrow \infty} \int_{-A}^A \cos \omega t d\omega \\ &= \lim_{A \rightarrow \infty} \frac{2 \sin At}{2\pi t} = \lim_{A \rightarrow \infty} \frac{\sin At}{\pi t} = \delta(t) \end{aligned}$$

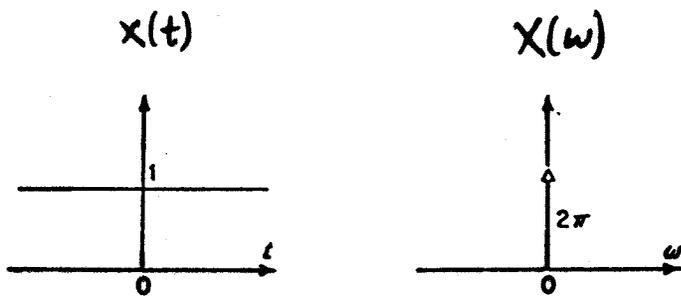
(as we ~~saw~~ saw on page ~~4.8~~ 4.10)

-So, $\delta(t) \longleftrightarrow 1$



EX: Applying the duality property to the preceding transform pair, we obtain

$$1 \xleftrightarrow{\mathcal{F}} 2\pi \delta(w)$$



EX: Applying the frequency shifting property to the preceding example, we get

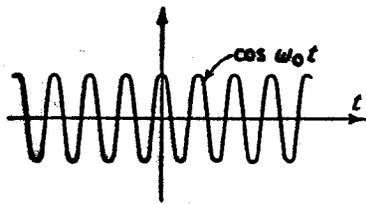
$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi \delta(w - \omega_0)$$

EX: $x(t) = \cos \omega_0 t$.

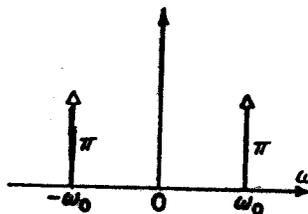
Using the linearity property and the result from the preceding example, we get

$$\begin{aligned} X(\omega) &= \mathcal{F}[\cos \omega_0 t] = \mathcal{F}\left[\frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})\right] \\ &= \frac{1}{2} \mathcal{F}[e^{j\omega_0 t}] + \frac{1}{2} \mathcal{F}[e^{-j\omega_0 t}] \\ &= \frac{1}{2} [2\pi \delta(\omega - \omega_0)] + \frac{1}{2} [2\pi \delta(\omega + \omega_0)] \\ &= \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \end{aligned}$$

$x(t)$

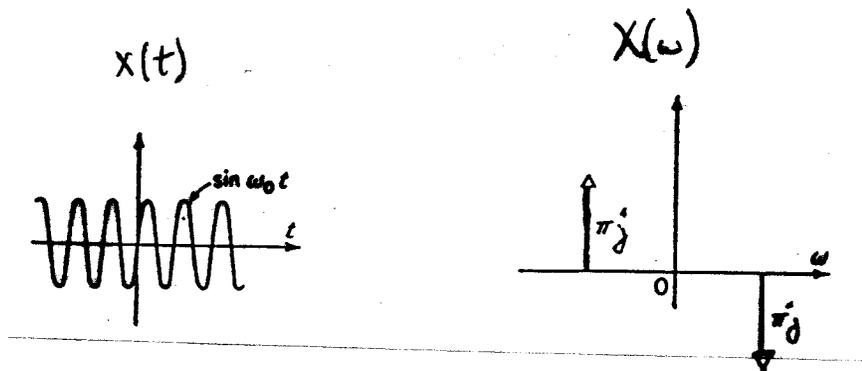


$X(\omega)$



EX: $x(t) = \sin \omega_0 t$.

$$\begin{aligned} X(\omega) &= \mathcal{F}[\sin \omega_0 t] = \mathcal{F}\left[\frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})\right] \\ &= \frac{1}{2j} \mathcal{F}[e^{j\omega_0 t}] - \frac{1}{2j} \mathcal{F}[e^{-j\omega_0 t}] \\ &= \frac{1}{2j} [2\pi \delta(\omega - \omega_0)] - \frac{1}{2j} [2\pi \delta(\omega + \omega_0)] \\ &= \frac{\pi}{j} \delta(\omega - \omega_0) - \frac{\pi}{j} \delta(\omega + \omega_0) \\ &= j\pi \delta(\omega + \omega_0) - j\pi \delta(\omega - \omega_0) \\ &= j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] \end{aligned}$$



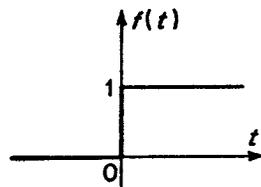
EX: $x(t) = u(t)$.

NOTE: $u(t) = \frac{1}{2} + \frac{1}{2} \operatorname{sgn} t$

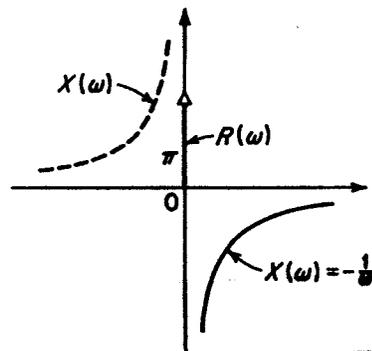
- Using the linearity property and the result of the preceding example, we have

$$\begin{aligned} U(\omega) &= \mathcal{F}[u(t)] = \mathcal{F}\left[\frac{1}{2} + \frac{1}{2} \operatorname{sgn} t\right] \\ &= \frac{1}{2} \mathcal{F}[1] + \frac{1}{2} \mathcal{F}[\operatorname{sgn} t] \\ &= \underline{\underline{\pi \delta(\omega) + \frac{1}{j\omega}}} \end{aligned}$$

$u(t)$



$U(\omega)$



EX: Time Integration Property:

$$\text{if } x_1(t) \xleftrightarrow{\mathcal{F}} X_1(\omega) \text{ and } x_2(t) = \int_{-\infty}^t x_1(\tau) d\tau,$$

$$\text{then } X_2(\omega) = X_1(\omega) \left[\pi \delta(\omega) + \frac{1}{j\omega} \right]$$

Proof:

$$\begin{aligned} x_2(t) &= \int_{-\infty}^t x_1(\tau) d\tau = \int_{-\infty}^{\infty} x_1(\tau) u(t-\tau) d\tau \\ &= x_1(t) * u(t). \end{aligned}$$

So, by the convolution property,

$$\begin{aligned} X_2(\omega) &= \mathcal{F}[x_2(t)] = X_1(\omega) U(\omega) \\ &= X_1(\omega) \left[\pi \delta(\omega) + \frac{1}{j\omega} \right]. \end{aligned}$$

QED

EX: $x(t) = u(t) \cos \omega_0 t$.

- Applying the frequency shifting property, we have

$$\begin{aligned} X(\omega) &= \mathcal{F}[u(t) \cos \omega_0 t] = \mathcal{F}\left[u(t) \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t}) \right] \\ &= \frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{1}{2j(\omega - \omega_0)} + \frac{1}{2j(\omega + \omega_0)} \\ &= \frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2} \end{aligned}$$

EX: $x(t) = u(t) \sin \omega_0 t$.

$$X(\omega) = \mathcal{F}[x(t)] = \mathcal{F}\left[u(t) \frac{1}{2j} (e^{j\omega_0 t} - e^{-j\omega_0 t})\right]$$

$$= \frac{\pi}{2j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{1}{-2(\omega - \omega_0)} - \frac{1}{-2(\omega + \omega_0)}$$

$$= \frac{\pi}{2j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega - \omega_0}{2(\omega^2 - \omega_0^2)} - \frac{\omega + \omega_0}{2(\omega^2 - \omega_0^2)}$$

$$= \frac{\pi}{2j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$$

- More transform pairs that we don't have time to derive:

$$e^{-t^2/2} \xleftrightarrow{\mathcal{F}} \sqrt{2\pi} e^{-\omega^2/2}$$

$$|t| \xleftrightarrow{\mathcal{F}} -\frac{2}{\omega^2}$$

$$\frac{d^n}{dt^n} \delta(t) \xleftrightarrow{\mathcal{F}} (j\omega)^n$$

$$t u(t) \xleftrightarrow{\mathcal{F}} j\pi \delta'(\omega) - \frac{1}{\omega^2}$$

$$t^n \xleftrightarrow{\mathcal{F}} 2\pi j^n \frac{d^n}{d\omega^n} \delta(\omega)$$

- When do Dirac deltas show up in the Fourier transform?

→ if $x(t)$ is a periodic signal, then $X(\omega)$ will be made up exclusively of Dirac deltas in ω . In other words, there will be nothing but "delta functions" in $X(\omega)$.

→ if $x(t)$ is a purely aperiodic signal, then there will not be any Dirac deltas in $X(\omega)$.

→ For a general signal $x(t)$ that has a periodic part $x_{\text{per}}(t)$ and an aperiodic part $x_{\text{aper}}(t)$, so that $x(t) = x_{\text{per}}(t) + x_{\text{aper}}(t)$,

⇒ The Fourier transform will be

$$X(\omega) = \mathcal{F}\{x(t)\}$$

$$= \mathcal{F}\{x_{\text{aper}}(t) + x_{\text{per}}(t)\}$$

$$= \underbrace{\mathcal{F}\{x_{\text{aper}}(t)\}}_{\text{no Dirac deltas}} + \underbrace{\mathcal{F}\{x_{\text{per}}(t)\}}_{\text{nothing but Dirac deltas}}$$

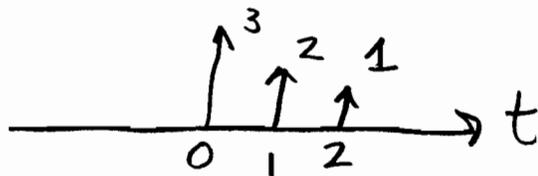
no Dirac deltas

nothing but Dirac deltas

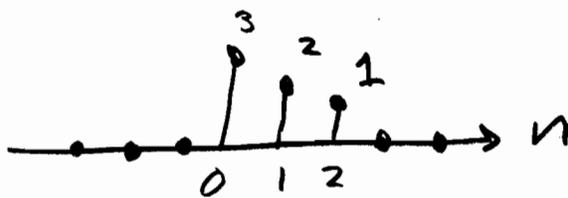
Duality Interpretation:

→ When a function is made up of nothing but Dirac deltas, you can think of it as a discrete function, where the samples are equal to the weights of the Dirac deltas.

EX: $x(t) = 3\delta(t) + 2\delta(t-1) + \delta(t-2)$



→ You can think of this as being equivalent to $x[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]$



FACT: A signal that is periodic in one domain is discrete in the other.

→ This is an important duality property of the Fourier transform.

→ It says this:

→ if $x(t)$ is periodic, then $X(\omega)$ is discrete (nothing but Dirac deltas).

→ if $x(t)$ is discrete (nothing but Dirac deltas), then $X(\omega)$ is periodic.

★ Another way to think of this:

→ In general, any signal $x(t)$ can be thought of as having a periodic part $x_{\text{per}}(t)$ and an aperiodic part $x_{\text{aper}}(t)$ [NOTE: either part might be zero in some cases].

⇒ The aperiodic part has a Fourier transform that is an ordinary function... no Dirac deltas (distributions).

⇒ The periodic part has a purely distributional Fourier transform.

⇒ Another interpretation of this is that: the periodic part of $x(t)$ doesn't have a Fourier transform that's an ordinary Riemann integral.

→ We can consider it to be a distributional integral,

→ Or we can write the periodic part of $x(t)$ in a Fourier series instead of a Fourier integral.

⇒ If we do this, the Fourier coefficients in the series will be numerically equal to the weights of the Dirac deltas in the distributional Fourier integral.

So, for a general $x(t)$ or $X(\omega)$, we can break it into parts like this:

- A part that is a plane old square-integrable function in one domain corresponds to a plane old square-integrable part in the other domain.
- A part that is periodic in one domain corresponds to a part in the other domain that is purely distributional... nothing but Dirac deltas. This distributional part can alternatively be thought of as a Fourier series.
- A part that is nothing but Dirac deltas in one domain corresponds to a purely periodic part in the other domain.

- Final comments on the

Discrete \longleftrightarrow Periodic
duality property of the Fourier transform;

① In chapter 5 we will take Fourier transforms of discrete signals $x[n]$.
 \Rightarrow we will see that these Fourier transforms are always periodic.

② When a signal is both periodic and discrete in one domain, it must also be both periodic and discrete in the other domain,

\Rightarrow This is called the discrete Fourier transform (DFT). It is not covered formally in the book.

\Rightarrow The fast Fourier transform (FFT) is a fast algorithm for computing the DFT.

FREQUENCY DOMAIN DESCRIPTION OF LSI SYSTEMS

4.3

- on page ~~4.1~~, we observed that the ~~set~~ signals $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$ are all eigenfunctions of any LSI continuous-time system.
- Given the eigenvalues for each ω , it was easy to write down the system output when the input was a complex exponential.
- on page ~~4.1~~^{4.11}, we also showed that, in a certain sense, the set $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$ is an "orthonormal" basis for a very large space of signals (up to that bothersome $\frac{1}{2\pi}$ factor!).
- We reasoned that, given the eigenvalues, we should be able to write an arbitrary signal $x(t)$ as a linear composition of this basis and then use the linearity of the system to easily find the output when $x(t)$ was the input.
- We wrote $H(\omega)$ for the set of eigenvalues.
- Specifically, on page ~~4.1~~^{4.3}, we saw that

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau,$$

where $h(t)$ was the system impulse response.

- Thus, we now see that the system frequency response, which gives the eigenvalues, is precisely the Fourier transform of the impulse response:

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = \mathcal{F}[h(t)]$$

$$H(\omega) \xleftrightarrow{\mathcal{F}} h(t)$$

- For an LSI system H with input $x(t)$ and output $y(t)$



the convolution property of the Fourier transform tells us that

$$Y(\omega) = X(\omega) H(\omega), \text{ where}$$

$Y(\omega)$: expresses the output $y(t)$ as a linear composition of complex sinusoids.

$X(\omega)$: expresses the input $x(t)$ as a linear composition of complex sinusoids.

$H(\omega)$: Gives the eigenvalues that tell how each individual complex sinusoid gets scaled as it passes through the system.

- So we are thinking of $x(t)$ as being a sum of terms like $X(\omega_1)e^{j\omega_1 t}$, $X(\omega_2)e^{j\omega_2 t}$, and $X(\omega_3)e^{j\omega_3 t}$,

i.e.,
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

- When $x(t)$ is input to an LTI system H , each one of these terms gets multiplied by an eigenvalue when it goes through the system:

Input term in $x(t)$ → Makes → output term in $y(t)$

$$X(\omega_1) e^{j\omega_1 t}$$

$$X(\omega_1) H(\omega_1) e^{j\omega_1 t}$$

$$X(\omega_2) e^{j\omega_2 t}$$

$$X(\omega_2) H(\omega_2) e^{j\omega_2 t}$$

$$X(\omega_3) e^{j\omega_3 t}$$

$$X(\omega_3) H(\omega_3) e^{j\omega_3 t}$$

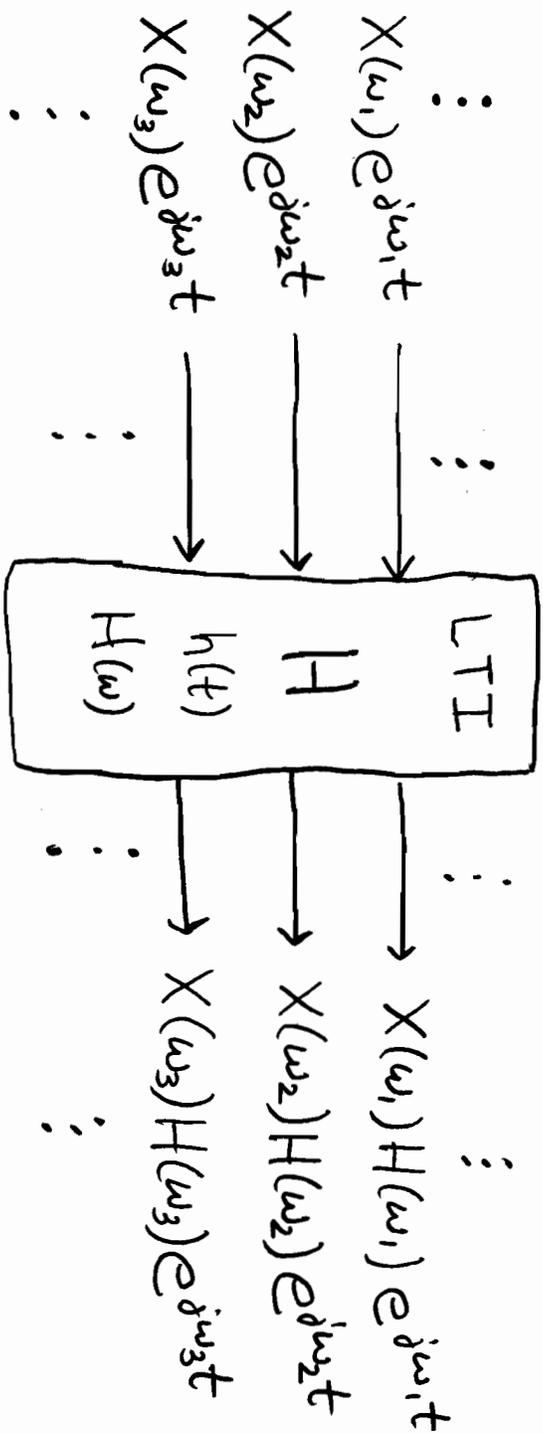
- And we are thinking of $y(t)$ as being a sum of all the terms of this form:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t} d\omega,$$

i.e.,
$$Y(\omega) = X(\omega) H(\omega).$$

IN PICTURES:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \longrightarrow \boxed{H} \longrightarrow y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega$$



$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t} d\omega$$

$$\Rightarrow Y(\omega) = X(\omega) H(\omega)$$

EX: H is an LTI system with impulse response

$$h(t) = e^{-at} u(t), \quad a \in \mathbb{R}, \quad a > 0.$$

The input is $x(t) = e^{-bt} u(t)$, $b \in \mathbb{R}$, $b > 0$,
 $b \neq a$.

Find $y(t)$.

Table: $X(\omega) = \frac{1}{b + j\omega}$

Table: $H(\omega) = \frac{1}{a + j\omega}$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

→ We use a partial fraction expansion to simplify $Y(\omega)$ into a sum of terms that are all in Table 4.2:

$$Y(\omega) = \frac{1}{(a + j\omega)(b + j\omega)} = \frac{A}{a + j\omega} + \frac{B}{b + j\omega} \quad (*)$$

→ We have to solve for the constants A and B in (*).

→ The equality in (*) has to hold for all choices of ω , including complex choices.



→ We can avoid the pitfalls of complex arithmetic by choosing $w = -j\theta$ for an arbitrary $\theta \in \mathbb{R}$.

→ EQ. (*) on PAGE 4.58 then becomes

$$\frac{1}{(a+\theta)(b+\theta)} = \frac{A}{a+\theta} + \frac{B}{b+\theta} \quad (**)$$

→ Solve (**) for A and B:

→ To get A, multiply both sides of (**) by $(a+\theta)$ and evaluate at $\theta = -a$:

$$\frac{1}{b+\theta} \Big|_{\theta=-a} = A + \frac{B(a+\theta)}{b+\theta} \Big|_{\theta=-a}$$

$$\boxed{\frac{1}{b-a} = A}$$

→ To get B, multiply both sides of (**) by $(b+\theta)$ and evaluate at $\theta = -b$:

$$\frac{1}{a+\theta} \Big|_{\theta=-b} = \frac{A(b+\theta)}{a+\theta} \Big|_{\theta=-b} + B$$

$$\boxed{\frac{1}{a-b} = B}$$

→ Now plug the solutions for A and B back into Eq. (*) on PAGE 4.58:

$$\begin{aligned} Y(\omega) &= \frac{A}{a+j\omega} + \frac{B}{b+j\omega} \\ &= \frac{1}{b-a} \frac{1}{a+j\omega} + \frac{1}{a-b} \frac{1}{b+j\omega} \\ &= \frac{1}{b-a} \left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega} \right] \end{aligned}$$

→ By table lookup and the linearity property,

$$y(t) = \mathcal{F}^{-1}\{Y(\omega)\} = \underline{\underline{\frac{1}{b-a} (e^{-at} - e^{-bt}) u(t)}}$$

EX: Same as before, except this time $a=b$.

$$h(t) = e^{-at} u(t), \quad x(t) = e^{-at} u(t), \quad a \in \mathbb{R}, \quad a > 0$$

$$\text{Table: } X(\omega) = H(\omega) = \frac{1}{a+j\omega}$$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(a+j\omega)^2}$$

→

→ This time $Y(\omega)$ is in the table, and we have immediately that

$$\underline{\underline{y(t) = te^{-at}u(t)}}$$

NOTE: In this case we can alternatively get the answer by applying the frequency differentiation property.

→ We already know that $e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega}$.
→ We observe that

$$Y(\omega) = \frac{1}{(a+j\omega)^2} = j \frac{d}{d\omega} \frac{1}{a+j\omega}$$

→ Applying the frequency differentiation property directly to the pair $e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega}$, we have

$$-j t e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega} \frac{1}{a+j\omega} = \frac{-j}{(a+j\omega)^2}$$

Multiply both sides by j :

$$t e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{(a+j\omega)^2} = Y(\omega)$$

$$\Rightarrow \underline{\underline{y(t) = te^{-at}u(t)}}$$

FILTERING INTERPRETATION

- In the frequency domain, the output of an LSI system is the product of the input with the system frequency response:

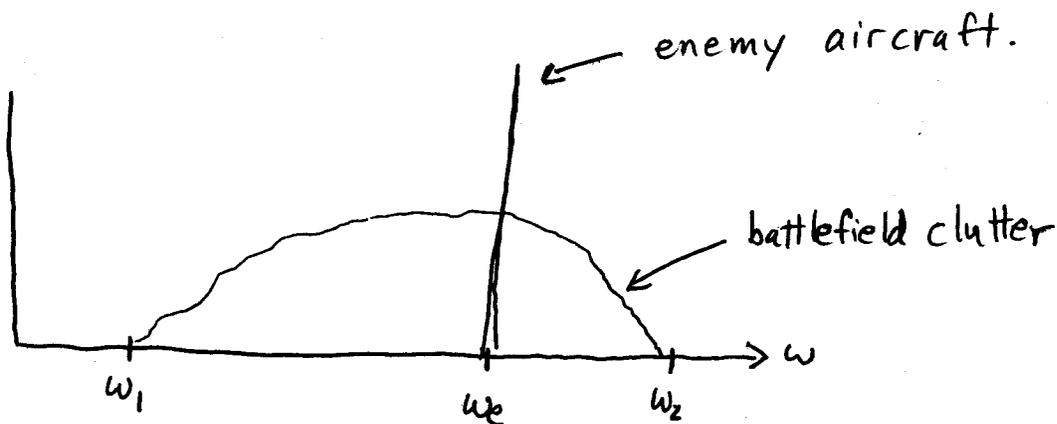
$$Y(\omega) = X(\omega) H(\omega).$$

- This can greatly simplify the analysis problem of figuring out what the output is for a given input (as compared to doing convolution in the time domain).
- It also simplifies the design problem.

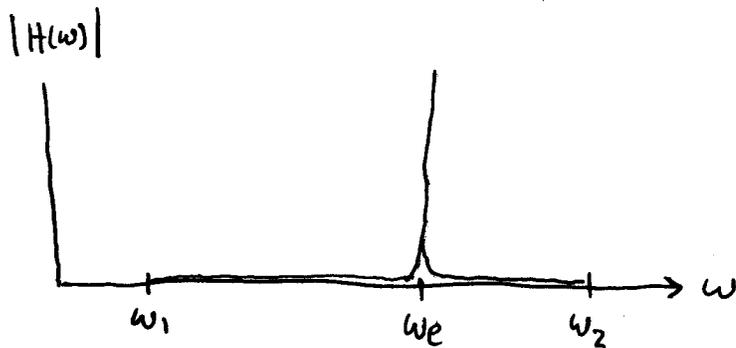
EX: On the battlefield, there is electromagnetic energy at frequencies ω between ω_1 and ω_2 .

- All enemy aircraft use radar processors with oscillators that emit energy at the frequency ω_e , where $\omega_1 < \omega_e < \omega_2$.

- Design an LSI system to detect the presence of enemy aircraft.



- Solution:



$$H(\omega) \approx \delta(\omega - \omega_e)$$

- use the inverse Fourier transform to design $h(t)$.

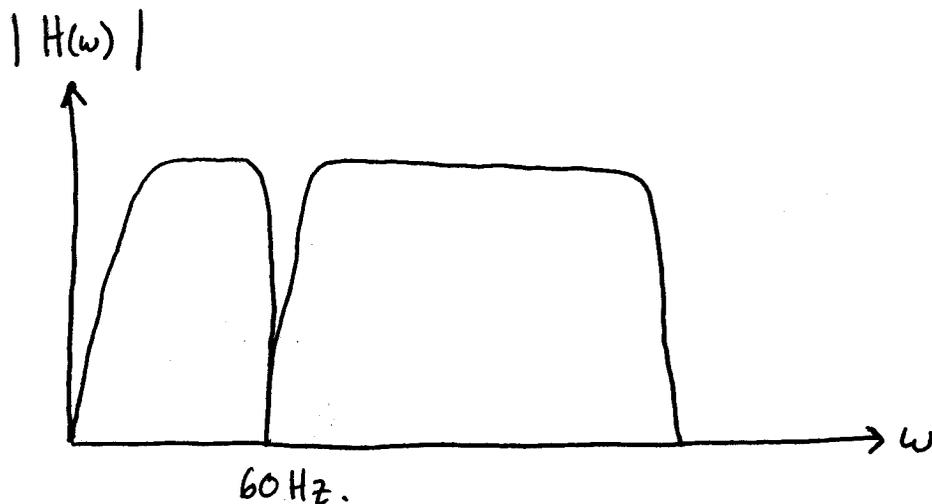
$$h(t) = \mathcal{F}^{-1}[\delta(\omega - \omega_e)] = e^{j\omega_e t}$$

- This type of approach is used in modern police radar detectors.

- The system output is large when the enemy is present.

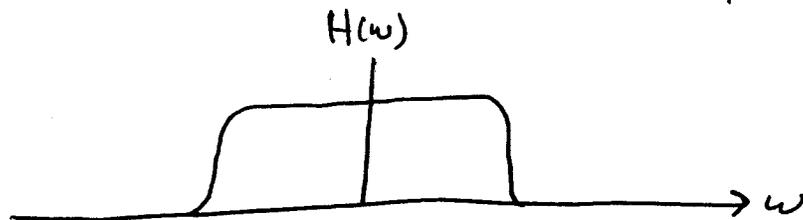
EX: A telephone line is being degraded by interference from a 60 Hz AC power supply.

- Design a filter to remove the noise.

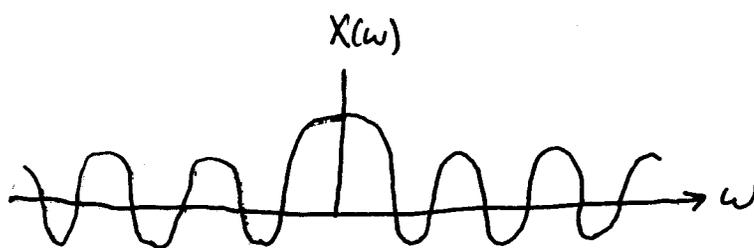


→ This is called a "notch filter".

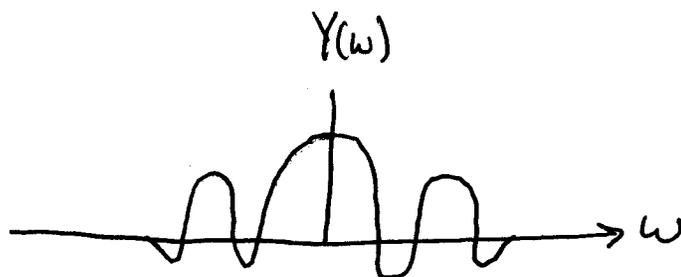
- Suppose H is an LSI system with frequency response



- Suppose $X(\omega)$ looks like:



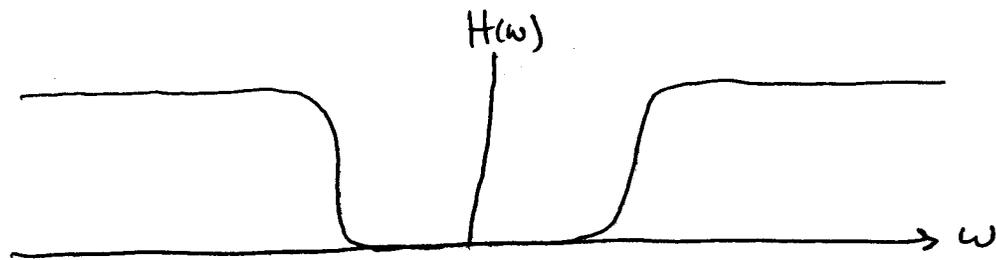
- Then $Y(\omega)$ looks like:



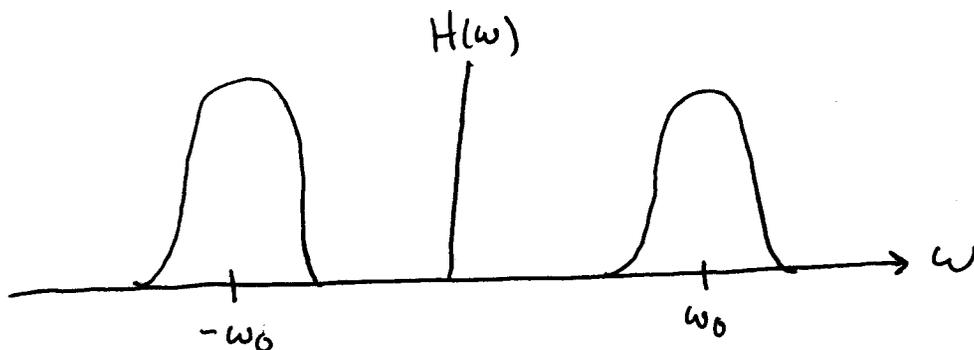
- The system passes low frequencies, but blocks high frequencies.

- This kind of LSI system is called a "Low-Pass filter".

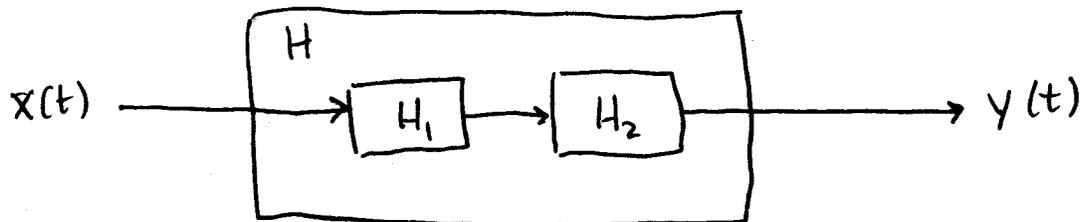
- Frequency response of a high-pass filter:



- Frequency response of a bandpass filter:



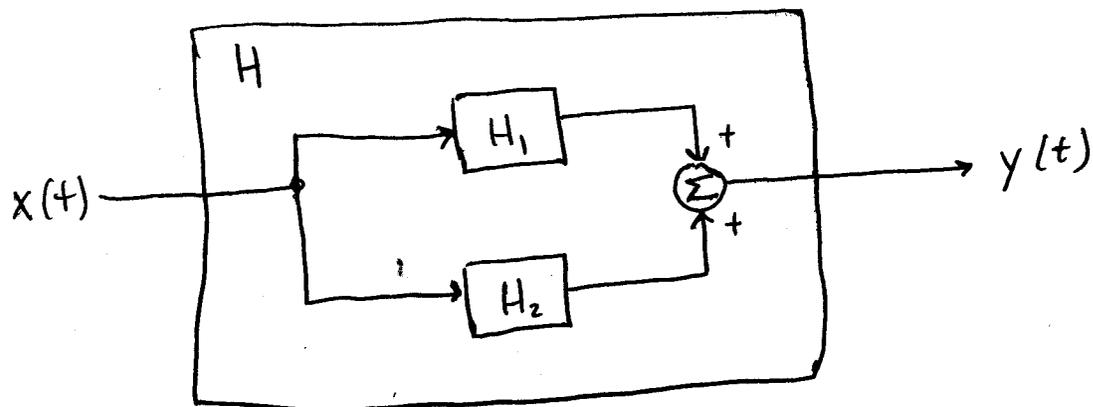
Series Connection of two LSI systems



$$h(t) = h_1(t) * h_2(t)$$

$$H(\omega) = H_1(\omega) H_2(\omega)$$

Parallel Connection of two LSI Systems

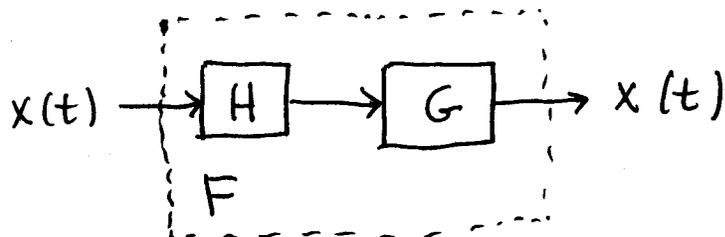


$$h(t) = h_1(t) + h_2(t)$$

$$H(\omega) = H_1(\omega) + H_2(\omega)$$

Invertible Systems

- Suppose G is the inverse system of an LSI system H :



- Then F is the identity system, and has impulse response $f(t) = \delta(t)$

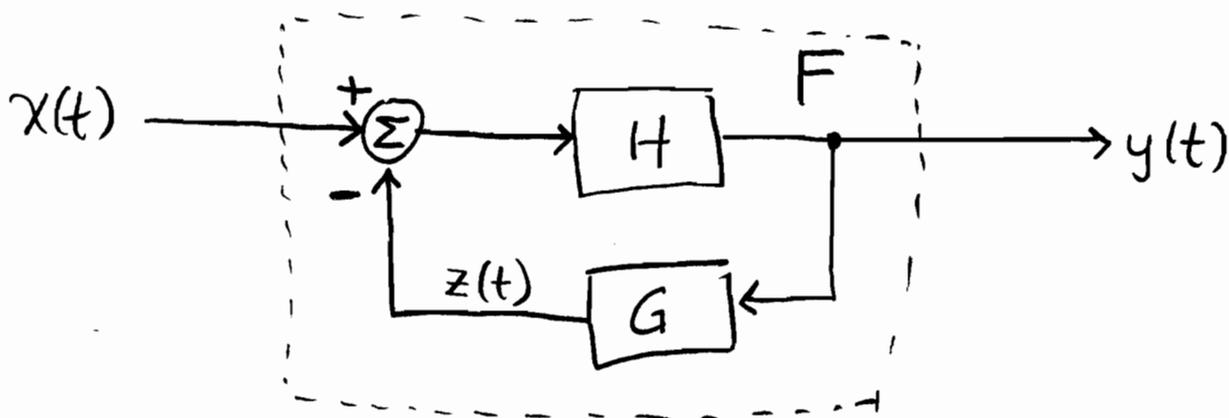
- Then $F(\omega) = \mathcal{F}[\delta(t)] = 1$.

- So $F(\omega) = H(\omega)G(\omega) = 1$.

- In other words, $G(\omega) = \frac{1}{H(\omega)}$

- Thus, an LSI system H has an inverse if $\frac{1}{H(\omega)}$ exists.

Feedback Connection



$$F(\omega) = \frac{H(\omega)}{1 + G(\omega)H(\omega)}$$

Proof :

$$z(t) = y(t) * g(t) \xleftrightarrow{\mathcal{F}} Z(\omega) = Y(\omega)G(\omega) \quad (*)$$

$$y(t) = [x(t) - z(t)] * h(t)$$

$$\xleftrightarrow{\mathcal{F}} Y(\omega) = [X(\omega) - Z(\omega)]H(\omega)$$

plug in (*): $Y(\omega) = [X(\omega) - Y(\omega)G(\omega)]H(\omega)$

$$Y(\omega) = X(\omega)H(\omega) - Y(\omega)G(\omega)H(\omega)$$

$$Y(\omega) + Y(\omega)G(\omega)H(\omega) = X(\omega)H(\omega)$$

$$Y(\omega) [1 + G(\omega)H(\omega)] = X(\omega)H(\omega)$$

$$F(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H(\omega)}{1 + G(\omega)H(\omega)} \quad \checkmark$$

EX: The input $x(t)$ and output $y(t)$ of an LTI System H are related by the differential equation

$$y''(t) + 4y'(t) + 3y(t) = x'(t) + 2x(t).$$

Find the impulse response $h(t)$.

- Taking Fourier transforms on both sides, we have

$$(j\omega)^2 Y(\omega) + 4j\omega Y(\omega) + 3Y(\omega) = j\omega X(\omega) + 2X(\omega)$$

$$[(j\omega)^2 + 4j\omega + 3] Y(\omega) = [j\omega + 2] X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} = \frac{j\omega + 2}{(j\omega + 3)(j\omega + 1)}$$

- Partial Fractions:

$$\frac{\theta + 2}{(\theta + 3)(\theta + 1)} = \frac{A}{\theta + 3} + \frac{B}{\theta + 1}$$

$$A = \frac{\theta + 2}{\theta + 1} \Big|_{\theta = -3} = \frac{-1}{-2} = \frac{1}{2}$$

$$B = \frac{\theta + 2}{\theta + 3} \Big|_{\theta = -1} = \frac{1}{2}$$

$$H(\omega) = \frac{1/2}{j\omega + 3} + \frac{1/2}{j\omega + 1}$$

$$h(t) = \frac{1}{2} e^{-3t} u(t) + \frac{1}{2} e^{-t} u(t)$$

EX: For the preceding example, find the output $y(t)$ when the input is given by $x(t) = e^{-t}u(t)$.

Table: $X(\omega) = \frac{1}{1+j\omega}$

$$Y(\omega) = X(\omega)H(\omega) = \frac{j\omega+2}{(j\omega+1)^2(j\omega+3)} = \frac{A}{j\omega+1} + \frac{B}{(j\omega+1)^2} + \frac{C}{j\omega+3}$$

Partial Fractions: $\frac{\theta+2}{(\theta+1)^2(\theta+3)} = \frac{A}{\theta+1} + \frac{B}{(\theta+1)^2} + \frac{C}{\theta+3}$

→ For C, multiply both sides by $\theta+3$ and evaluate at $\theta=-3$:

$$\frac{\theta+2}{(\theta+1)^2} \Big|_{\theta=-3} = \underbrace{\frac{(\theta+3)A}{(\theta+1)} \Big|_{\theta=-3}}_{\text{ZERO}} + \underbrace{\frac{(\theta+3)B}{(\theta+1)^2} \Big|_{\theta=-3}}_{\text{ZERO}} + C$$

$$C = \frac{-3+2}{(-3+1)^2} = \frac{-1}{(-2)^2} = -\frac{1}{4}$$

→ For B, multiply both sides by $(\theta+1)^2$ and evaluate at $\theta=-1$:

$$\frac{\theta+2}{\theta+3} \Big|_{\theta=-1} = \underbrace{\frac{(\theta+1)A}{\theta+3} \Big|_{\theta=-1}}_{\text{ZERO}} + B + \underbrace{\frac{(\theta+1)^2 C}{\theta+3} \Big|_{\theta=-1}}_{\text{ZERO}}$$

$$B = \frac{-1+2}{-1+3} = \frac{1}{2}$$

→

→ strategy for A:

① multiply both sides by $(\theta+1)^2$.

→ The left side will have no $(\theta+1)$ in the denominator.

→ The right side will have a "B" term with no θ left, a "C" term that is quadratic in $(\theta+1)$, and an "A" term that is linear in θ .

② Take $\frac{d}{d\theta}$ on both sides.

→ The left side will still have no $(\theta+1)$ in the denominator.

→ On the right side,

→ The "B" term will be gone.

→ The "C" term will still have $(\theta+1)$ in the numerator.

→ The "A" term will no longer have θ .

③ Evaluate both sides at $\theta = -1$ to get a solution for A.

⇒ THIS STRATEGY ALWAYS WORKS.

→ Find A: multiply both sides by $(\theta+1)^2$, take $\frac{d}{d\theta}$, and evaluate at $\theta = -1$:

$$\left[\frac{d}{d\theta} \frac{(\theta+2)(\theta+1)^2}{(\theta+1)^2(\theta+3)} \right]_{\theta=-1} = \left[\frac{d}{d\theta} (\theta+1)A \right]_{\theta=-1} + \frac{d}{d\theta} B + \left[\frac{d}{d\theta} \frac{(\theta+1)^2 C}{\theta+3} \right]_{\theta=-1}$$

→ To avoid using the "quotient rule", I rewrite this in a way that the "product rule" can be used:

$$\left[\frac{d}{d\theta} (\theta+2)(\theta+3)^{-1} \right]_{\theta=-1} = \left[\frac{d}{d\theta} (\theta+1)A \right]_{\theta=-1} + 0 + \left[(\theta+1)^2(\theta+3)^{-1} C \right]_{\theta=-1}$$

$$\left[(\theta+2)(-1)(\theta+3)^{-2} + 1(\theta+3)^{-1} \right]_{\theta=-1} = A + \left[(\theta+1)^2(-1)(\theta+3)^{-2} + 2(\theta+1)(\theta+3)^{-1} \right]_{\theta=-1} C$$

$$\left[(-1+2)(-1)(-1+3)^{-2} + (-1+3)^{-1} \right] = A + [0 + 0] C$$

$$\left[-\frac{1}{2^2} + \frac{1}{2} \right] = A$$

$$-\frac{1}{4} + \frac{1}{2} = A$$

$$A = \frac{1}{4}$$

- So we have $A = \frac{1}{4}$, $B = \frac{1}{2}$, $C = -\frac{1}{4}$

$$Y(s) = \frac{1/4}{1+j\omega} + \frac{1/2}{(1+j\omega)^2} - \frac{1/4}{3+j\omega}$$

Table: $y(t) = \frac{1}{4}e^{-t}u(t) + \frac{1}{2}te^{-t}u(t) - \frac{1}{4}e^{-3t}u(t)$

NOTE: Partial fractions is an "Ad Hoc" technique. There is more than one way to do it.

The method presented here is called the "Heaviside coverup method."

It may not be the shortest solution, but it always works and it always works the same way.

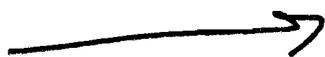
- The general form of the input/output equation for an LTI system governed by a constant coefficients ordinary linear differential equation is:

$$\sum_{k=0}^N a_k \frac{d^k}{dt^k} y(t) = \sum_{m=0}^M b_m \frac{d^m}{dt^m} x(t),$$

where the a_k 's and b_m 's are constants.

→ In other words, a linear combination of the derivatives of the output is equal to a linear combination of the derivatives of the input.

→ By taking Fourier transforms on both sides, we can derive a general solution for the frequency response of any system of this type.



$$\mathcal{F}\left\{\sum_{k=0}^N a_k \frac{d^k}{dt^k} y(t)\right\} = \mathcal{F}\left\{\sum_{m=0}^M b_m \frac{d^m}{dt^m} x(t)\right\}$$

$$\sum_{k=0}^N a_k \mathcal{F}\left\{\frac{d^k}{dt^k} y(t)\right\} = \sum_{m=0}^M b_m \mathcal{F}\left\{\frac{d^m}{dt^m} x(t)\right\}$$

$$\sum_{k=0}^N a_k (j\omega)^k Y(\omega) = \sum_{m=0}^M b_m (j\omega)^m X(\omega)$$

$$Y(\omega) \sum_{k=0}^N a_k (j\omega)^k = X(\omega) \sum_{m=0}^M b_m (j\omega)^m$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{m=0}^M b_m (j\omega)^m}{\sum_{k=0}^N a_k (j\omega)^k}$$

\Rightarrow We are done with Chapter 4.