5-10. Frequency Domain Analysis of Discrete-Time Signals and Systems.

- when we studied discrete-time signals and systems in the time domain, we wrote the signals as linear combinations of the basis consisting of all integer translates of the Kronecker delta:

$$\{\delta[n-k]\}_{k\in\mathbb{Z}}.$$

- Using the properties of LSI systems, we concluded their outputs are the convolutions of their unit pulse responses with their inputs.

- In Chapters 5 and 10, we will use bases made up of discrete-time complex exponential signals.

- This will enable us to develop frequency domain analysis techniques for discrete-time signals and systems.

Eigenfunction Interpretation, Frequency Response, And Transfer Function.

- Let H be a discrete-time LSI system with unit pulse response $h[n]$.

- When the system input is $x[n]$, the output $y[n]$ is given by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]h[k].$$

- For some particular fixed value of ω, suppose the input is $x[n] = e^{j\omega n}$.

- Then $$y[n] = \sum_{k \in \mathbb{Z}} e^{j\omega(n-k)} h[k]$$

$$= \underbrace{e^{j\omega n}}_{\text{The input}} \underbrace{\sum_{k \in \mathbb{Z}} e^{-j\omega k} h[k]}_{\text{A complex number for any fixed } \omega.}$$

$\Longrightarrow$ ANY complex exponential is an eigenfunction of any discrete-time LSI system.

- The set of eigenvalues $\sum_{k \in \mathbb{Z}} h[k] e^{-j\omega k}$ depend on ω and on the unit pulse response $h[k]$.

- For any given system H, we will write the eigenvalues together for all values of ω as

$$H(e^{j\omega}) = \sum_{k \in \mathbb{Z}} h[k] e^{-j\omega k}$$

$\Rightarrow$ $H(e^{j\omega})$ is called the "<u>frequency</u> <u>response</u>" of the discrete-time LSI system H.

<u>NOTE:</u> The domain of $h[k]$ and $x[k]$ and $y[k]$ is $\mathbb{Z}$.

$\rightarrow$ The domain of $H(e^{j\omega})$ is $\mathbb{R}$. ☆☆

$\Rightarrow$ So $H(e^{j\omega})$ is kind of like a continuous-time signal.

- Now consider a discrete-time LSI system H with unit pulse response $h[n]$.

- This time, for some particular fixed $z \in \mathbb{C}$, let the input be $x[n] = z^n$.

- The output $y[n]$ is given by

$$y[n] = x[n] * h[n] = \sum_{k \in \mathbb{Z}} z^{n-k} h[k]$$

$$= \underbrace{z^n}_{\text{The input}} \underbrace{\sum_{k \in \mathbb{Z}} h[k] z^{-k}}_{\text{a complex number for any fixed } z.}$$

→ Thus, z^n, $z \in \mathbb{C}$, is an eigenfunction of any discrete-time LSI system.

- The associated eigenvalue depends on both the system and the particular value of z.

- We can write the eigenvalues together for all values of z as

$$H(z) = \sum_{k \in \mathbb{Z}} h[k] z^{-k}$$

- $H(z)$ is called the "transfer function" of the discrete-time LSI system H.

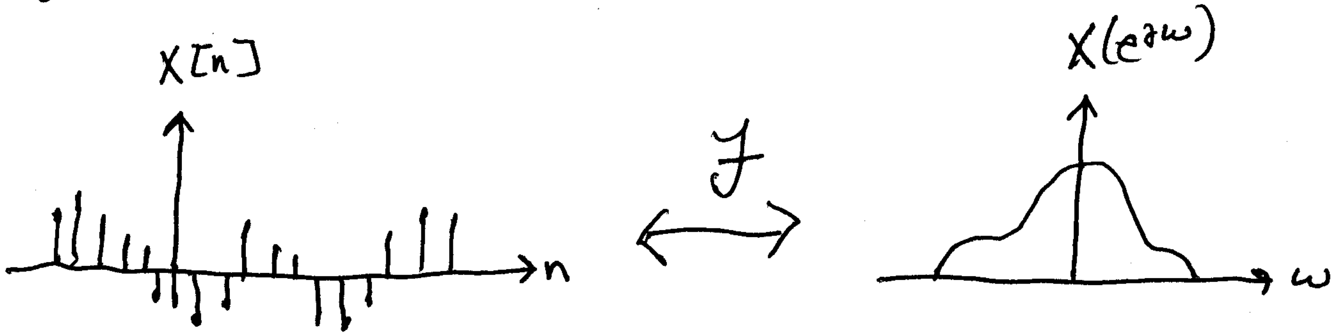
Fourier Transform of a Sequence

- For any LSI discrete-time system H, it is easy to tell how the system responds to an input of the form $x[n] = e^{j\omega n}$.
- Thus, for an arbitrary input $x[n]$, it will be useful to write $x[n]$ as a linear composition of the basis $\{e^{j\omega n}\}_{\omega \in \mathbb{R}}$.
- We do this as we always do, by taking inner products between the signal $x[n]$ and the basis functions:

$$\langle x[n], e^{j\omega n} \rangle = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}$$

$$\equiv X(e^{j\omega}).$$

- $X(e^{j\omega})$ is called the "Fourier Transform" of $x[n]$.



NOTE: The frequency response $H(e^{j\omega})$ is the Fourier transform of the impulse response $h[n]$:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}.$$

→ The book calls $X(e^{j\omega})$ the "discrete-time" Fourier transform.

→ This is not the same as the "discrete Fourier Transform", or DFT, that you may have heard of.

- On pages 1.51 - 1.52 of the notes, we saw that

  → all possible discrete cosines $\cos \omega_0 n$ can be created using ω_0 s/ in the range $0 \leq \omega_0 \leq \pi$.

  → all possible discrete sines $\sin \omega_0 n$ can be created using ω_0 s/ in the range $-\pi < \omega_0 \leq \pi$.

⇒ Thus, all possible complex exponentials $e^{j\omega_0 n}$ can be created using frequencies ω_0 in the range $-\pi < \omega_0 \leq \pi$.

  → for ω_0 <u>outside</u> this range, the signal $e^{j\omega_0 n}$ takes the <u>same</u> <u>values</u> at every n as another signal $e^{j\tilde{\omega}_0 n}$, where $\tilde{\omega}_0 \in [-\pi, \pi]$.

- In fact, if $\omega_0 \in \mathbb{R}$, then

$$\cos[(\omega_0 + 2\pi k)n] = \cos \omega_0 n,$$

$$\sin[(\omega_0 + 2\pi k)n] = \sin \omega_0 n,$$

$$e^{j(\omega_0 + 2\pi k)n} = e^{j\omega_0 n},$$

for any $k \in \mathbb{Z}$.

- So, we intuitively expect that $X(e^{j\omega})$ should be 2π-periodic in ω.

- This is, in fact, the case.

<u>FACT</u>: if $x[n]$ is a discrete-time signal (real or complex valued), then the Fourier transform $X(e^{j\omega})$ is 2π-periodic in ω:

$$X(e^{j(\omega + m2\pi)}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j(\omega + 2\pi m)k}$$

$$= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} e^{-j2\pi mk}$$

$$= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} (1)^{mk}$$

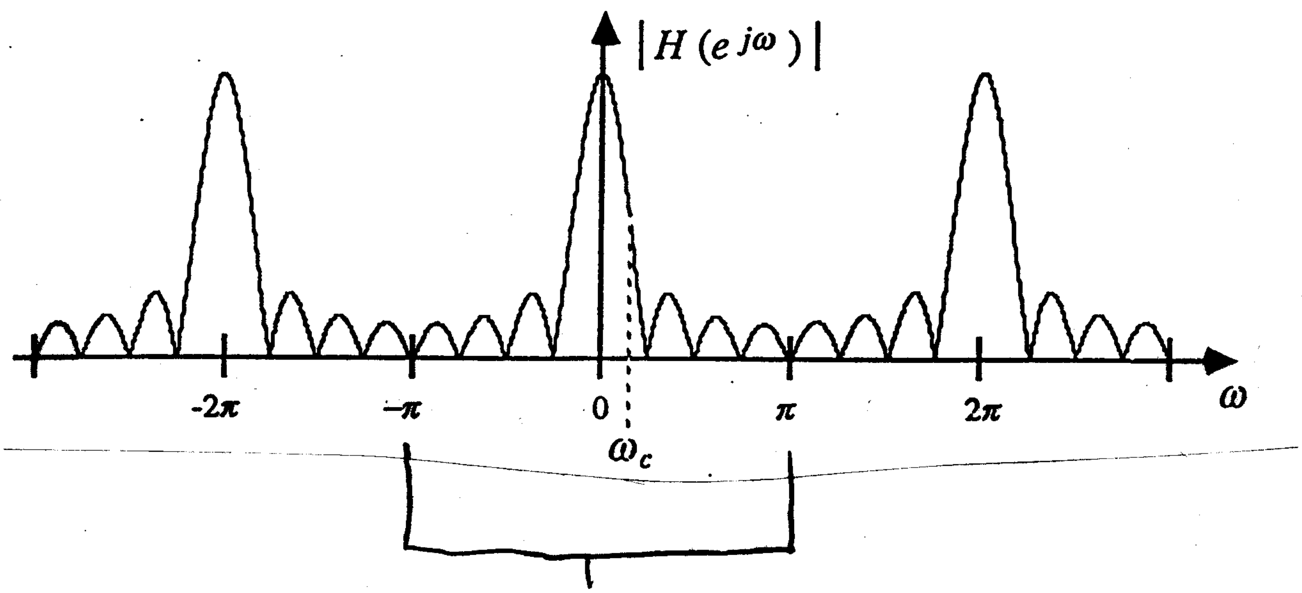
$$= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} = X(e^{j\omega}) \checkmark$$

(where $m \in \mathbb{Z}$).

→ Usually, when we graph $X(e^{j\omega})$ we only graph one period, either $\omega \in [-\pi, \pi]$ or $\omega \in [0, 2\pi]$.

- But always keep it in your mind that the Fourier transform of a sequence is 2π-periodic.

- EX: Frequency response of a low-pass filter:



Fundamental Period.

- When does $X(e^{j\omega})$ exist (converge) ?

FACT: if $\sum_{k \in \mathbb{Z}} |x[k]| < \infty$, then $X(e^{j\omega})$ exists.

FACT: if $\left[\sum_{k \in \mathbb{Z}} |x[k]|^2\right]^{1/2} < \infty$, then $X(e^{j\omega})$ exists.

NOTE: if $\sum_{k \in \mathbb{Z}} |x[k]| \to \infty$ and $\sum_{k \in \mathbb{Z}} |x[k]|^2 \to \infty$,

then $X(e^{j\omega})$ might or might not exist.

- For each ω, $X(e^{j\omega})$ is generally a complex number.
- Thus, we can write $X(e^{j\omega})$ in polar form:

$$X(e^{j\omega}) = |X(e^{j\omega})| \exp\{j \arg X(e^{j\omega})\}$$

where

$$|X(e^{j\omega})| = \left\{ \left[\mathrm{Re}\{X(e^{j\omega})\}\right]^2 + \left[\mathrm{Im}\{X(e^{j\omega})\}\right]^2 \right\}^{1/2}$$

$$\arg X(e^{j\omega}) = \arctan \frac{\mathrm{Im}[X(e^{j\omega})]}{\mathrm{Re}[X(e^{j\omega})]}.$$

→ $|X(e^{j\omega})|$ is called the <u>magnitude spectrum</u> of the signal $x[n]$.

→ $\arg X(e^{j\omega})$ is called the <u>spectral phase</u> of $x[n]$.

<u>Note</u>: the book writes $\measuredangle X(e^{j\omega})$ instead of $\arg X(e^{j\omega})$.

- We write $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$ to indicate that $x[n]$ and $X(e^{j\omega})$ are a Fourier transform pair.

- Study examples 5.1, 5.2, and 5.3 on pages 362-366. These are straightforward examples where sum formulas are used to compute discrete-time Fourier transforms from the definition.

Inversion of Discrete-Time Fourier Transform

- Once we know $X(e^{j\omega})$, the coordinates of $x[n]$ with respect to the basis $\{e^{j\omega n}\}_{\omega \in [-\pi,\pi]}$, we can write $x[n]$ as an uncountable linear composition of the basis over any period of length 2π:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi}^{4\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega.$$

$\Rightarrow$ This is called the inverse Fourier transform of $X(e^{j\omega})$.

- Recall that the continuous-time inverse F.T. converged to the midpoints of discontinuities of $x(t)$.

- Since continuity is not a property of discrete-time signals, this does not occur in discrete time.

$\Rightarrow$ if $X(e^{j\omega}) = \mathcal{F}\left[x[n]\right] = \sum_{k \in \mathbb{Z}} x[k] e^{-j\omega k}$ exists,

then $\mathcal{F}^{-1}\left[X(e^{j\omega})\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega = x[n] \ \forall n.$

EX: $x[n] = \delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{otherwise}. \end{cases}$

$$X(e^{j\omega}) = \sum_{n \in \mathbb{Z}} \delta[n] e^{-j\omega n} = 1 \cdot e^{-j0} = 1$$

Inversion: $\mathcal{F}^{-1}\{1\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega n} d\omega = \frac{1}{j2\pi n} e^{j\omega n}\Big|_{-\pi}^{\pi}$

$$= \frac{1}{n\pi} \frac{e^{j\pi n} - e^{-j\pi n}}{2j} = \frac{\sin n\pi}{n\pi}.$$

when $n \neq 0$, $\frac{\sin n\pi}{n\pi} = \frac{0}{n\pi} = 0$.

$\longrightarrow$

when $n=0$, $\quad \dfrac{\sin n\pi}{n\pi} = \dfrac{0}{0}$ (indeterminate).

Use L^Hospital's rule:

$$\lim_{n\to 0} \frac{\sin n\pi}{n\pi} = \lim_{a\to 0} \frac{\sin a\pi}{a\pi} = \lim_{a\to 0} \frac{\frac{d}{da}\sin a\pi}{\frac{d}{da} a\pi}$$

$$= \lim_{a\to 0} \frac{\pi \cos a\pi}{\pi} = \frac{\pi}{\pi} = 1.$$

So, $\mathcal{F}^{-1}\{1\} = \begin{cases} 1, & n=0 \\ 0, & \text{otherwise} \end{cases} = \delta[n].$

$\Rightarrow$ Thus, $\quad \delta[n] \xleftrightarrow{\mathcal{F}} 1.$

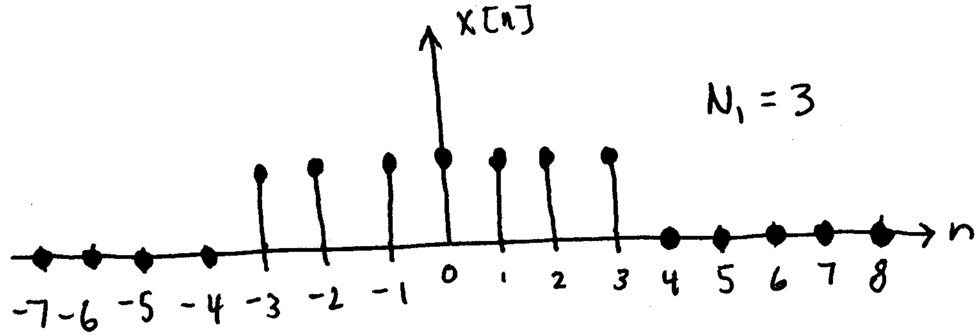
<u>NOTE</u>: the following sum formula will be useful:

$$\sum_{n=N_1}^{N_2} \alpha^n = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \quad \begin{matrix} \alpha \neq 1 \\ N_2 \geq N_1. \end{matrix}$$

$\Rightarrow$ Add this formula to your sum formula sheet.

EX:

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & \text{otherwise.} \end{cases}$$



- This signal is called the discrete-time boxcar, or "comb function".

$$X(e^{j\omega}) = \sum_{n \in \mathbb{Z}} x[n] e^{-j\omega n} = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{-1} e^{-j\omega n} + 1 + \sum_{n=1}^{N_1} e^{-j\omega n}$$

$$\sum_{n=1}^{N_1} (e^{-j\omega})^n = \frac{e^{-j\omega} - e^{-j\omega(N_1+1)}}{1 - e^{-j\omega}}$$

$$\sum_{n=-N_1}^{-1} e^{-j\omega n} = \sum_{n=1}^{N_1} e^{j\omega n} = \frac{e^{j\omega} - e^{j\omega(N_1+1)}}{1 - e^{j\omega}}$$

$\longrightarrow$

$$X(e^{j\omega}) = 1 + \frac{e^{-j\omega} - e^{-j\omega(N_1+1)}}{1-e^{-j\omega}} + \frac{e^{j\omega} - e^{j\omega(N_1+1)}}{1-e^{j\omega}}$$

$$= \frac{(1-e^{j\omega})(1-e^{-j\omega})}{(1-e^{j\omega})(1-e^{-j\omega})} + \frac{(1-e^{j\omega})\left[e^{-j\omega} - e^{-j\omega(N_1+1)}\right]}{(1-e^{j\omega})(1-e^{-j\omega})}$$

$$+ \frac{(1-e^{-j\omega})\left[e^{j\omega} - e^{j\omega(N_1+1)}\right]}{(1-e^{-j\omega})(1-e^{j\omega})}$$

$$= \frac{1}{(1-e^{j\omega})(1-e^{-j\omega})}\Big[1-e^{-j\omega}-e^{j\omega}+1+e^{-j\omega}-e^{-j\omega(N_1+1)}-1$$
$$+e^{-j\omega N_1}+e^{j\omega}-e^{j\omega(N_1+1)}-1+e^{j\omega N_1}\Big]$$

$$= \frac{1}{1-e^{-j\omega}-e^{j\omega}+1}\left\{\left[e^{-j\omega N_1}+e^{j\omega N_1}\right]-\left[e^{-j\omega(N_1+1)}+e^{j\omega(N_1+1)}\right]\right\}$$

$$= \frac{1}{2-(e^{j\omega}+e^{-j\omega})}\left\{2\cos N_1\omega - 2\cos(N_1+1)\omega\right\}$$

$$= \frac{2\cos N_1\omega - 2\cos(N_1+1)\omega}{2-2\cos\omega} = \frac{\cos N_1\omega - \cos(N_1+1)\omega}{1-\cos\omega}$$

– Apply trig identity $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$ to denominator:

$$1-\cos\omega = 2\sin^2\frac{\omega}{2}$$

- Apply trig identity $\cos A - \cos B = 2\sin\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(B-A)\right]$ to numerator:

$$\underbrace{\cos N_1\omega}_{A} - \underbrace{\cos(N_1+1)\omega}_{B} = 2\sin\left[\tfrac{1}{2}(2N_1+1)\omega\right]\sin\left[\tfrac{1}{2}\omega\right]$$

$$= 2\sin\left[\omega(N_1+\tfrac{1}{2})\right]\sin\frac{\omega}{2}$$

$$X(e^{j\omega}) = \frac{2\sin\left[\omega(N_1+\frac{1}{2})\right]\sin\frac{\omega}{2}}{2\sin^2\frac{\omega}{2}} = \frac{\sin\left[\omega(N_1+\frac{1}{2})\right]}{\sin\frac{\omega}{2}}$$

- Thus,

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & \text{otherwise} \end{cases} \xleftrightarrow{\mathcal{F}} \frac{\sin\left[\omega(N_1+\frac{1}{2})\right]}{\sin\frac{\omega}{2}}.$$

<u>EX</u>:

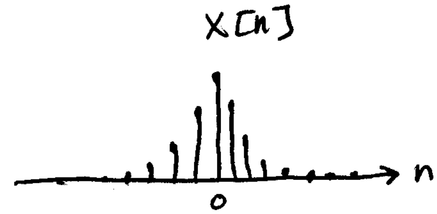
$$x[n] = a^n u[n], \quad |a| < 1.$$

$$X(e^{j\omega}) = \sum_{n\in\mathbb{Z}} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}, \quad \text{since } |ae^{-j\omega}| < 1.$$

$$a^n u[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-ae^{j\omega}}, \quad |a| < 1.$$

EX: $x[n] = a^{|n|}, \quad |a| < 1.$



$$X(e^{j\omega}) = \sum_{n \in \mathbb{Z}} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$$= \sum_{m=1}^{\infty} a^{m} e^{j\omega m} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty} (ae^{j\omega})^{n} + \sum_{n=0}^{\infty} (ae^{-j\omega})^{n}$$

$$= \frac{ae^{j\omega} - \cancel{(ae^{j\omega})^{\infty}}^{\to 0}}{1 - ae^{j\omega}} + \frac{1}{1 - ae^{-j\omega}}$$

$$= \frac{(1 - ae^{-j\omega})\, ae^{j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} + \frac{1 - ae^{j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})}$$

$$= \frac{ae^{j\omega} - a^2 + 1 - ae^{j\omega}}{1 - ae^{j\omega} - ae^{-j\omega} + a^2} = \underline{\underline{\frac{1 - a^2}{1 - 2a\cos\omega + a^2}}}$$

$$a^{|n|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1 - a^2}{1 - 2a\cos\omega + a^2}, \quad |a| < 1.$$

Proof of Inversion Integral

$$\mathcal{F}^{-1}\left\{\mathcal{F}\left[x[n]\right]\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}\{x[n]\} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{ \sum_{k\in\mathbb{Z}} x[k] e^{-j\omega k} \right\} e^{j\omega n}\, d\omega$$

$$= \sum_{k\in\mathbb{Z}} x[k] \left\{ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-k)}\, d\omega \right\}. \quad (*)$$

Now, $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-k)}\, d\omega = \frac{1}{2\pi} \frac{1}{j(n-k)} e^{j\omega(n-k)} \Big|_{-\pi}^{\pi}$

$$= \frac{1}{\pi(n-k)} \frac{e^{j\pi(n-k)} - e^{-j\pi(n-k)}}{2j}$$

$$= \frac{\sin \pi(n-k)}{\pi(n-k)} = \delta[n-k],$$ as we showed on page 5-10.13 using L^Hospital's rule.

So,

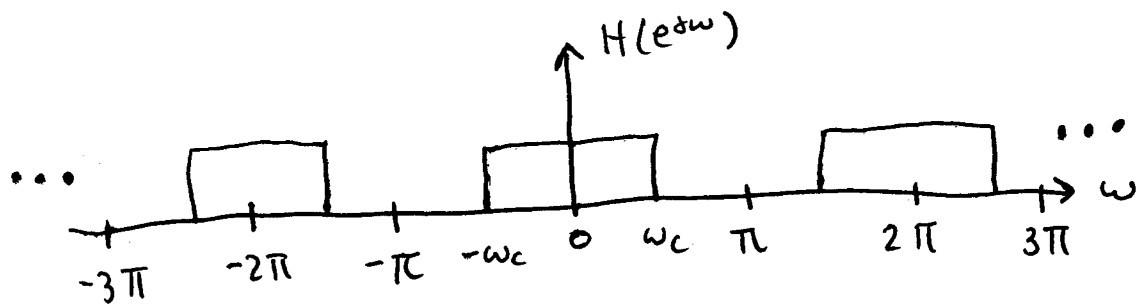
$$\mathcal{F}^{-1}\left\{\mathcal{F}\left[x[n]\right]\right\} = (*) = \sum_{k\in\mathbb{Z}} x[k]\,\delta[n-k]$$

$$= x[n] * \delta[n]$$

$$= x[n] \checkmark$$

EX: The fundamental period of $H(e^{j\omega})$ is given by

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi. \end{cases}$$



$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{1}{j2\pi n} e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = \frac{1}{\pi n}\,\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}$$

$$= \underline{\underline{\frac{\sin \omega_c n}{\pi n}}}$$

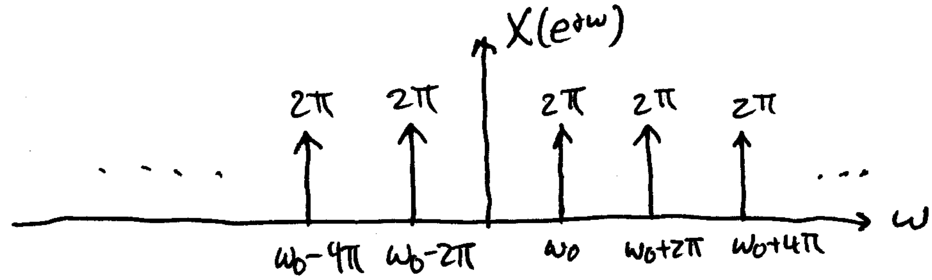
Discrete-Time Distributional Fourier Transforms

- On page 367, the book gives the Fourier transform pair

$$e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi \underbrace{\delta(\omega - \omega_0 - 2\pi l)}_{\text{Dirac Delta}}$$

without deriving it.

- In this case, the graph of $X(e^{j\omega})$ looks like



which is 2π-periodic, as it must be.

- With our distribution theory, we can establish this transform pair directly.

→ Assume $|\omega_0| < \pi$.

<u>Inverse Transform:</u>

$$\mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{\ell=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0-2\pi\ell)\,e^{j\omega n}\,d\omega$$

$$= \cancel{\frac{1}{2\pi}}\sum_{\ell=-\infty}^{\infty}\int_{-\pi}^{\pi}\cancel{2\pi}\,\delta(\omega-\omega_0-2\pi\ell)\,e^{j\omega n}\,d\omega$$

$$= \underbrace{\int_{-\pi}^{\pi}\delta(\omega-\omega_0)\,e^{j\omega n}\,d\omega}_{\ell=0} + \underbrace{\sum_{\ell=1}^{\infty}\int_{-\pi}^{\pi}\delta(\omega-\omega_0-2\pi\ell)\,e^{j\omega n}\,d\omega}_{0}$$

$$+ \underbrace{\sum_{\ell=-\infty}^{-1}\int_{-\pi}^{\pi}\delta(\omega-\omega_0-2\pi\ell)\,e^{j\omega n}\,d\omega}_{0}$$

→

$$= \int_{-\pi}^{\pi} \delta(\omega-\omega_0) e^{j\omega n} \, d\omega = \int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j\omega n} \, d\omega$$

$$= e^{j\omega_0 n} = x[n] \checkmark$$

<u>Forward Transform:</u>

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega n} = \lim_{M\to\infty} \sum_{n=-M}^{M} e^{j(\omega_0-\omega)n}$$

→ Let $\theta = \omega - \omega_0$

$$= \lim_{M\to\infty} \sum_{n=-M}^{M} e^{-j\theta n} = \lim_{M\to\infty} \frac{\sin[\theta(M+\frac{1}{2})]}{\sin\frac{\theta}{2}}$$ (as we showed on page 5-10.16)

$$= \lim_{M\to\infty} \frac{\sin[(\omega-\omega_0)(M+\frac{1}{2})]}{\sin\left[\frac{\omega-\omega_0}{2}\right]} \qquad (*)$$

→ Eq. (*) is clearly 2π-periodic in ω.

→ Let $\tilde{X}(e^{j\omega})$ be the fundamental period:

$$\tilde{X}(e^{j\omega}) = \lim_{M\to\infty} \frac{\sin[(\omega-\omega_0)(M+\frac{1}{2})]}{\sin\left[\frac{\omega-\omega_0}{2}\right]}, \quad -\pi < \omega-\omega_0 < \pi$$

→ Then $$X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} \tilde{X}(e^{j(\omega-2\pi\ell)}). \qquad (**)$$

⟶

- So, in the interval $-\pi < \omega - \omega_0 < \pi$,

$$\tilde{X}(e^{j\omega}) = \lim_{M \to \infty} \frac{\sin[(\omega-\omega_0)(M+\frac{1}{2})]}{\omega-\omega_0} \frac{\omega-\omega_0}{\sin(\frac{\omega-\omega_0}{2})}$$

$$= \frac{\omega-\omega_0}{\sin(\frac{\omega-\omega_0}{2})} \lim_{M \to \infty} \frac{\sin[(\omega-\omega_0)(M+\frac{1}{2})]}{\omega-\omega_0}$$

$\Rightarrow$ Recall from page 4.10; $\lim_{A \to \infty} \frac{\sin At}{\pi t} = \delta(t)$.

So, $$\lim_{M \to \infty} \frac{\sin[(M+\frac{1}{2})(\omega-\omega_0)]}{\omega-\omega_0} = \pi\delta(\omega-\omega_0)$$

So, $$\tilde{X}(e^{j\omega}) = \frac{\omega-\omega_0}{\sin(\frac{\omega-\omega_0}{2})} \pi\,\delta(\omega-\omega_0).$$

$\Rightarrow$ But, $x(t)\delta(t) = x(0)\delta(t)$ as a distribution,

so, $$\tilde{X}(e^{j\omega}) = \left.\frac{\omega-\omega_0}{\sin(\frac{\omega-\omega_0}{2})}\right|_{\omega=\omega_0} \pi\,\delta(\omega-\omega_0).$$

$\rightarrow$ Use L'Hospital's Rule:

$$\left.\frac{\omega-\omega_0}{\sin(\frac{\omega-\omega_0}{2})}\right|_{\omega=\omega_0} = \lim_{\omega \to \omega_0} \frac{\omega-\omega_0}{\sin(\frac{\omega-\omega_0}{2})} = \lim_{\omega \to \omega_0} \frac{\frac{d}{d\omega}\,\omega-\omega_0}{\frac{d}{d\omega}\sin(\frac{\omega-\omega_0}{2})}$$

$$= \lim_{\omega \to \omega_0} \frac{1}{\frac{1}{2}\cos(\frac{\omega-\omega_0}{2})} = 2.$$

$\Rightarrow$ So $\tilde{X}(e^{j\omega}) = 2\pi\,\delta(\omega-\omega_0)$

$\longrightarrow$

- Plug this result for $\tilde{X}(e^{j\omega})$ into (**) on page 5-10.21:

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \tilde{X}(e^{j(\omega-2\pi l)})$$

$$= \sum_{l=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0-2\pi l) \quad \checkmark$$

This rigorously establishes the transform pair

$$e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l).$$

→ Later, we will show that the discrete-time Fourier transform is linear. For now, we will assume that this is true.

- Then, using the result for $e^{j\omega_0 n}$ above, we obtain

$$\mathcal{F}[\cos\omega_0 n] = \mathcal{F}\left[\tfrac{1}{2}e^{j\omega_0 n} + \tfrac{1}{2}e^{-j\omega_0 n}\right]$$

$$= \tfrac{1}{2}\mathcal{F}[e^{j\omega_0 n}] + \tfrac{1}{2}\mathcal{F}[e^{-j\omega_0 n}]$$

$$= \tfrac{1}{2}\sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l) + \tfrac{1}{2}\sum_{l=-\infty}^{\infty} 2\pi\delta(\omega+\omega_0-2\pi l)$$

$$= \pi\sum_{l=-\infty}^{\infty}\left[\delta(\omega-\omega_0-2\pi l) + \delta(\omega+\omega_0-2\pi l)\right].$$

- Thus,

$$\cos\omega_0 n \overset{\mathcal{F}}{\longleftrightarrow} \pi\sum_{l=-\infty}^{\infty}\left\{\delta(\omega-\omega_0-2\pi l) + \delta(\omega+\omega_0-2\pi l)\right\}$$

- Similarly,

$$\mathcal{F}[\sin\omega_0 n] = \frac{1}{2j}\mathcal{F}[e^{j\omega_0 n}] - \frac{1}{2j}\mathcal{F}[e^{-j\omega_0 n}]$$

$$= \frac{1}{2j}\sum_{\ell=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi\ell) - \frac{1}{2j}\sum_{\ell=-\infty}^{\infty} 2\pi\delta(\omega+\omega_0-2\pi\ell)$$

$$= \frac{\pi}{j}\sum_{\ell=-\infty}^{\infty}\left\{\delta(\omega-\omega_0-2\pi\ell) - \delta(\omega+\omega_0-2\pi\ell)\right\}.$$

Thus,

$$\sin\omega_0 n \xleftrightarrow{\mathcal{F}} \frac{\pi}{j}\sum_{\ell=-\infty}^{\infty}\left\{\delta(\omega-\omega_0-2\pi\ell) - \delta(\omega+\omega_0-2\pi\ell)\right\}.$$

The Bilateral Z-transform

- The bilateral Z-transform is like a discrete-time version of the bilateral Laplace transform.
- Suppose $x[n]$ is a "bad" signal, so that the Fourier transform

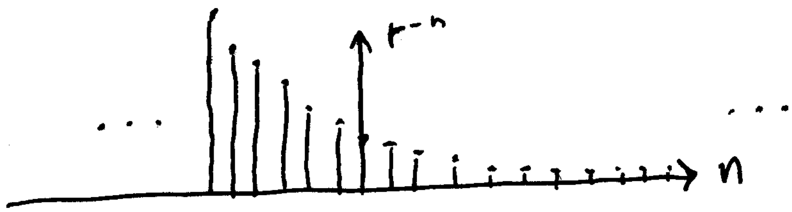
$$X(e^{j\omega}) = \sum_{k\in\mathbb{Z}} x[k]e^{-j\omega k} \quad \underline{\text{diverges}}.$$

- Perhaps we can "fix things up" like we did with the Laplace transform.

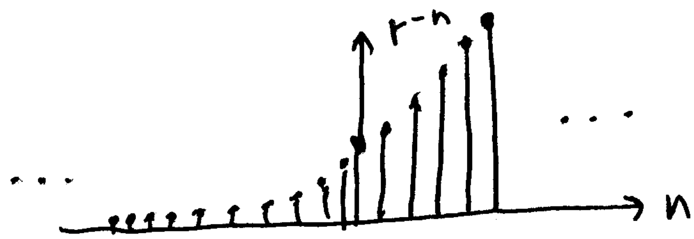
- Suppose we multiply $x[n]$ by a real-valued exponential sequence r^{-n}, where $r \in \mathbb{R}$ and $r \geq 0$,

$\longrightarrow$

- If $|r| \geq 1$, then r^{-n} looks like



- If $|r| < 1$, then r^{-n} looks like



→ For a sequence $x[n]$ with a Fourier transform that diverges, there may be certain values of r such that the Fourier transform of the "fixed up" signal

$$x[n]r^{-n}$$

<u>does</u> converge.

- The Fourier transform of $x[n]r^{-n}$ is

$$\sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n} \qquad (*).$$

- Let $z = re^{j\omega}$. Then z is a complex number.

  → As r and ω range over $\mathbb{R}$, z ranges over $\mathbb{C}$.

→

- In terms of z, (*) on page 5-10.25 becomes

$$\mathcal{F}[x[n]r^{-n}] = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] (re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

- This is the Z-transform of $x[n]$:

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n \in \mathbb{Z}} x[n] z^{-n}.$$

- We write $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ to indicate that $x[n]$ and $X(z)$ are a Z-transform pair.

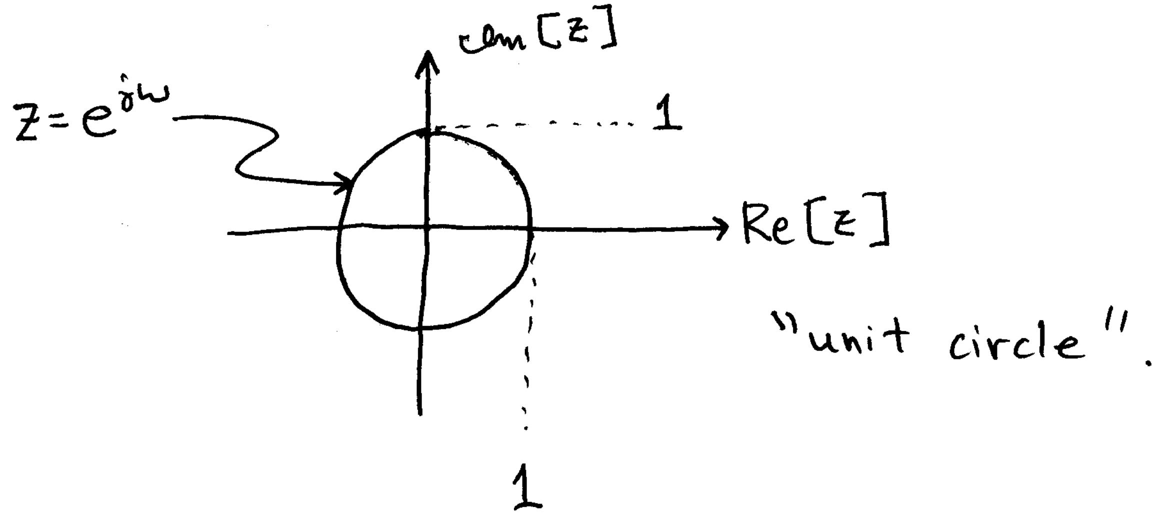
- The relationship between the Z-transform $X(z)$ and the Fourier transform $X(e^{j\omega})$ is

$$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}}.$$

- FACT: $e^{j\omega}$ is a complex number with unit magnitude.

$\rightarrow$ Thus, as ω ranges through $\mathbb{R}$, $e^{j\omega}$ maps out the unit circle in the complex z-plane.

$\longrightarrow$



$\Rightarrow$ The Fourier transform $X(e^{j\omega})$ of a sequence $x[n]$ is equal to the Z-transform $X(z)$ evaluated on the unit circle.

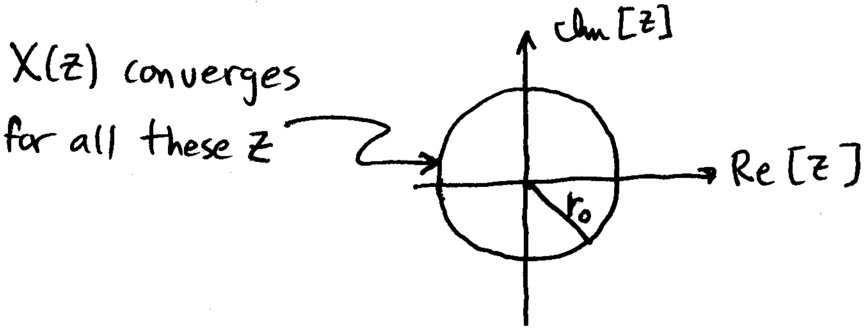
- For a general signal $x[n]$, $X(z)$ will converge for some values of $z = re^{j\omega}$ (Fixer-upper helped), and it will diverge for others (Fixer-upper hurt).
- The set of z for which the Z-transform converges is called the Region of Convergence, or ROC, of $X(z)$.
- Suppose $X(z) = \mathcal{Z}\{x[n]\}$ converges for a particular value z_0 of z, with $r_0 = |z_0|$.

  $\rightarrow$ This means that $\mathcal{F}\{x[n]r_0^{-n}\}$ converges for all $\omega \in \mathbb{R}$:

$$\left| \sum_{n \in \mathbb{Z}} x[n] r_0^{-n} e^{-j\omega n} \right| < \infty \qquad \forall \omega \in \mathbb{R}.$$

$\longrightarrow$

- Thus, if $X(z)$ converges for $z = z_0$, then $X(z)$ also converges for all z lying on the circle $|z| = r_0$:



→ Thus, the ROC of $X(z)$ always consists of circles or rings in the z-plane.

FACT: The ROC of $X(z)$ does not contain any poles.

EX: $x[n] = a^n u[n]$, $a \in \mathbb{R}$.

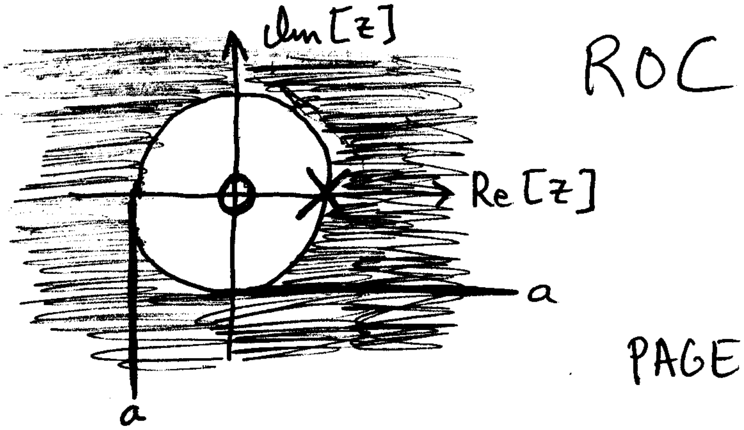
$$X(z) = \sum_{n \in \mathbb{Z}} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}}, \quad |a| < |z|$$

$$= \frac{z}{z-a}, \quad |a| < |z|.$$

→ If $|z| \leq |a|$, the transform diverges.

→ $X(z)$ has a first-order zero at $z=0$ and a first-order pole at $z=a$.



EX: $x[n] = -a^n u[-n-1]$, $a \in \mathbb{R}$.

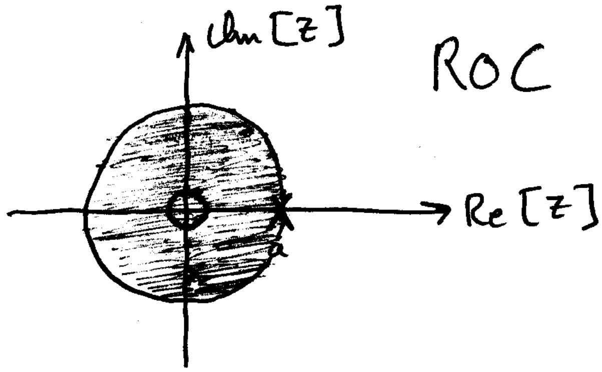
$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^{n} = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{1-a^{-1}z}, \quad |z| < |a|$$

$$= \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

$$= \frac{z}{z-a}, \quad |z| < |a|$$



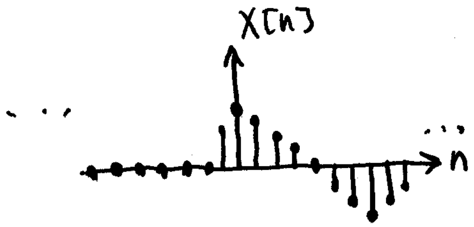
$\Rightarrow$ This transform has the same functional form as the one in the last example.

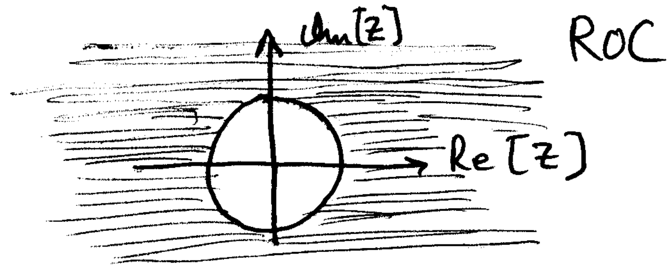
$\rightarrow$ Only the ROC's are different.

NOTE: if $X(z) = X_1(z) + X_2(z)$, then the ROC of $X(z)$ is the intersection of the ROC's of $X_1(z)$ and $X_2(z)$.

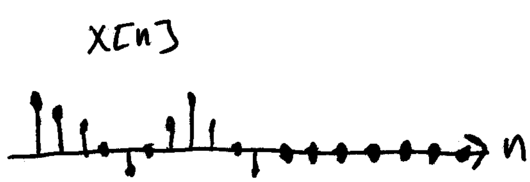
- **FACT**: if $x[n]$ is of finite duration, then the ROC of $X(z)$ is the entire z-plane, except possibly the points $z=0$ and $z=\infty$.

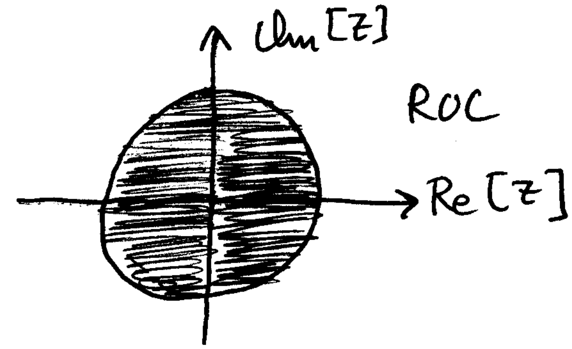
- **FACT**: If $x[n]$ is right-sided and the ROC of $X(z)$ contains the circle $|z|=|z_0|$, then the ROC also contains all finite z with $|z|>|z_0|$:



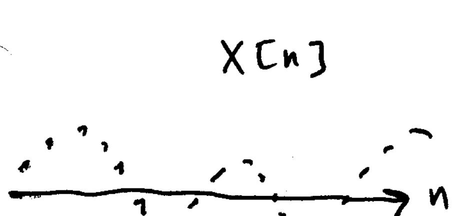


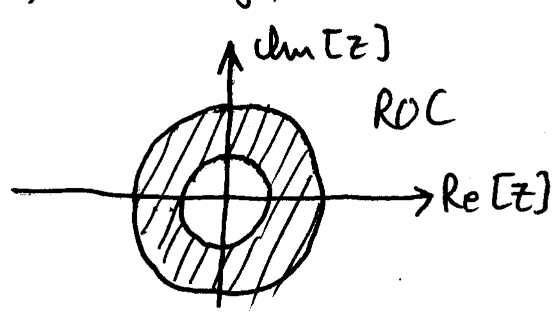
- **FACT**: If $x[n]$ is left-sided and the ROC of $X(z)$ contains the circle $|z|=|z_0|$, then the ROC also contains all nonzero z with $|z|<|z_0|$:





- **FACT**: if $x[n]$ is two-sided, then the ROC of $X(z)$ is generally an annulus, or ring, in the z-plane:





- **FACT**: if $x[n]$ is right-sided and $X(z)$ is rational, then the ROC of $X(z)$ is the set of all finite z outside the outermost pole of $X(z)$.

  → If $x[n]$ is also causal, $x[n]=0 \ \forall n<0$, then the ROC also contains the point $z=\infty$.

- **FACT**: if $x[n]$ is left-sided and $X(z)$ is rational, then the ROC of $X(z)$ is the set of all nonzero z inside the innermost pole of $X(z)$.

  → If $x[n]=0 \ \forall n>0$, then the ROC also contains the point $z=0$.

EX:

$$x[n] = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise.} \end{cases} \qquad a>0.$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

→ Since $x[n]$ is of finite duration, the ROC is the whole z-plane, except possibly the points $z=0$ and $z=\infty$.

→

→ since $x[n] = 0 \ \forall n < 0$, the ROC includes $z = \infty$ in this case.

→ since $x[n]$ is nonzero for some positive values of n,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

contains some negative powers of z, so the point $z=0$ is not in the ROC in this case.

⇒ Recall: $X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$

→ There is a pole of order $N-1$ at $z=0$.

→ The roots of $z^N - a^N$ are $z_k = a e^{j2\pi k/N}$, $k = 0, 1, \ldots, N-1$.

→ There is a pole at $z=a$ that is cancelled by the zero $z_k = a e^{j2\pi k/N}$ when $k=0$.

→ The remaining $N-1$ roots z_k of the numerator for $k = 1, 2, \ldots, N-1$ are zeros of $X(z)$. They all have magnitude a, but each one has a different angle.

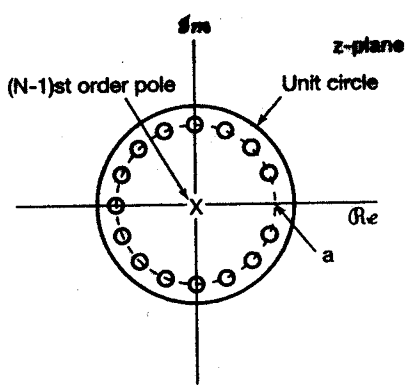


Figure 10.9 Pole-zero pattern for Example 10.6 with $N = 16$ and $0 < a < 1$. The region of convergence for this example consists of all values of z except $z = 0$.

Inversion of Bilateral Z-transform

- $X(z)$ is the Fourier transform of $x[n]r^{-n}$, where $z = re^{j\omega}$.

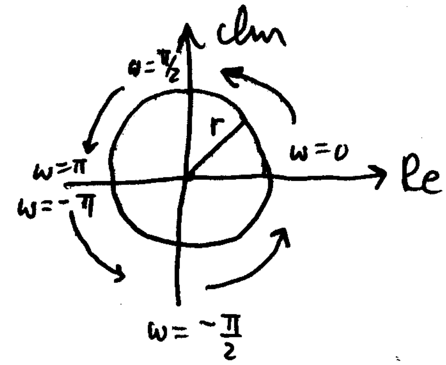
- So, $x[n]r^{-n} = \mathcal{F}^{-1}\{X(z)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(z)e^{j\omega n}\,d\omega$

→ Then
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(z)\,r^n e^{j\omega n}\,d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(z)\,(re^{j\omega})^n\,d\omega$$

- This says that $x[n]$ can be recovered from $X(z)$ by integrating $X(z)(re^{j\omega})^n$ with respect to ω from $-\pi$ to π.

→ This is true for any r such that the circle $|z| = r$ is in the ROC of $X(z)$.

→ Integrating ω from $-\pi$ to π is equivalent to integrating $X(z)(re^{j\omega})^n = X(z)z^n$ around the circle $|z| = r$, moving in a counterclockwise direction:



- Let us change the variable of integration from ω to z, where it is understood that $|z|=r$ will remain <u>fixed</u> throughout the integration.

- Since $z = re^{j\omega}$,

$$\frac{dz}{d\omega} = \frac{d}{d\omega} re^{j\omega} = r \frac{d}{d\omega} e^{j\omega} = jre^{j\omega} = jz$$

- So, $\frac{dz}{d\omega} = jz$; $dz = jz\,d\omega$; $d\omega = \frac{1}{j} z^{-1} dz$.

- Plugging this into $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(z)(re^{j\omega})^n d\omega$, we get

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz,$$

which is the <u>formal</u> definition of the inverse Z-transform.

→ The contour integration can be taken around any circle $|z|=r$ is the ROC of $X(z)$.

- To prove this, we need Cauchy's integral theorem, which says that

$$\frac{1}{2\pi j} \oint_C z^{-m} dz = \delta[m-1] \quad \text{(kronecker delta)}.$$

→

Proof: $$\frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz = \frac{1}{2\pi j}\oint_C \left(\sum_{k\in\mathbb{Z}} x[k]z^{-k}\right)z^{n-1}dz$$

$$= \frac{1}{2\pi j}\oint_C \sum_{k\in\mathbb{Z}} x[k]z^{n-k-1}dz$$

$$= \frac{1}{2\pi j}\sum_{k\in\mathbb{Z}} x[k]\oint_C z^{n-k-1}dz$$

$$= \sum_{k\in\mathbb{Z}} x[k]\left\{\frac{1}{2\pi j}\oint_C z^{-(k+1-n)}dz\right\}$$

$$= \sum_{k\in\mathbb{Z}} x[k]\delta[k-n] = x[n]*\delta[n] = x[n] \checkmark$$

- In practice, we rarely evaluate inverse z-transforms using the definition.
- Instead, we use a table like table 10.2 on page 776 of the book.
- For transforms that are not in the table, we use partial fractions and/or properties like those in table 10.1 on page 775 of the book to get terms that are in the table.
- Alternatively, we can recognize $X(z)$ as the power series $X(z) = \sum_{n\in\mathbb{Z}} x[n]z^{-n}$, and then just "pick off" the values of $x[n]$.

EX:

$$X(z) = z^2(1-\tfrac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})\ ; \text{ ROC: all } z \text{ except } z=0 \text{ and } z=\infty.$$

$$= (1-\tfrac{1}{2}z^{-1})(z+1)(z-1)$$

$$= (1-\tfrac{1}{2}z^{-1})(z^2-1)$$

$$= z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}$$

$$= 1z^2 - \tfrac{1}{2}z^1 - 1z^0 + \tfrac{1}{2}z^{-1}$$

$x[-2]=1$ $\quad$ $x[-1]=-\frac{1}{2}$ $\quad$ $x[0]=-1$ $\quad$ $x[1]=\frac{1}{2}$

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1].$$

EX: $X(z) = \log(1+az^{-1}), \quad |z|>|a|.$

→ Expand $X(z)$ in a Taylor Series:

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n}{n} z^{-n}$$

$$x[n] = \begin{cases} (-1)^{n+1}\dfrac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

- If the other methods fail, you can also try inverting $X(z)$ by long division:

EX: $X(z) = \dfrac{1}{1-az^{-1}}$, $|z| > |a|$.

→ Since the ROC is the exterior of a circle, $x[n]$ is a right-sided sequence.

→ Thus, we want to perform the division so that we get decreasing powers of z in the quotient, since

$$X(z) = \sum_{n \in \mathbb{Z}} x[n] z^{-n}.$$

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} \; \ldots \\ \hline 1-az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2z^{-2}} \\ & \qquad a^2z^{-2} \\ & \qquad \underline{a^2z^{-2} - a^3z^{-3}} \\ & \qquad\qquad a^3z^{-3} \ldots \end{array}$$

So $X(z) = \underset{\substack{\uparrow \\ x[0]}}{1z^0} + \underset{\substack{\uparrow \\ x[1]}}{az^{-1}} + \underset{\substack{\uparrow \\ x[2]}}{a^2z^{-2}} + \underset{\substack{\uparrow \\ x[3]}}{a^3z^{-3}} + \ldots$ $\ldots$

$$\underline{x[n] = a^n u[n]}.$$

EX: $X(z) = \dfrac{1}{1-az^{-1}}$, $|z| < |a|$.

→ Since the ROC is the interior of a circle, $x[n]$ is left-sided.

→ Thus, we want to perform the division so that we get increasing powers of z in the quotient.

$$\frac{1}{1-az^{-1}} = \frac{z}{z-a} = \frac{z}{-a+z}$$

$$\begin{array}{r l}
 & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 \cdots \\ \hline
-a+z \,\big) & z \\
 & z - a^{-1}z^2 \\ \hline
 & \quad a^{-1}z^2 \\
 & \quad a^{-1}z^2 - a^{-2}z^3 \\ \hline
 & \qquad a^{-2}z^3 \\
 & \qquad a^{-2}z^3 - a^{-3}z^4 \\ \hline
 & \qquad\quad a^{-3}z^4 \cdots
\end{array}$$

$$X(z) = \underbrace{-a^{-1}z^1}_{x[-1]} \underbrace{-a^{-2}z^2}_{x[-2]} \underbrace{-a^{-3}z^3}_{x[-3]} \underbrace{-a^{-4}z^4}_{x[-4]} - \cdots \quad \cdots$$

$$\underline{x[n] = -a^n u[-n-1]}$$

→ Study the examples in sections 10.1 – 10.3 of the book.

Discrete-time Fourier and Z-transform Properties

Linearity

if $x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$

and $x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega})$

Then $\mathcal{F}\{a_1 x_1[n] + a_2 x_2[n]\}$

$$= a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

if $x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)$, ROC R_1

and $x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z)$, ROC R_2

Then $\mathcal{Z}\{a_1 x_1[n] + a_2 x_2[n]\}$

$$= a_1 X_1(z) + a_2 X_2(z),$$

with ROC $R_1 \cap R_2$.

Note: The ROC may be larger if some poles cancel.

Time Shifting

if $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$

Then $x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$

if $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, ROC R

Then $x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z)$, with ROC R, except for possibly the points $z=0$ and $z=\pm\infty$.

Frequency Shifting

(Fourier Transform Only)

if $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$

Then $e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$

Z-domain Scaling

(Z-transform only)

if $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, ROC R

Then $z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{z}{z_0}\right)$,

with ROC $|z_0|R$.

Conjugation

if $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$

Then $x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$

if $x[n] \xleftrightarrow{\mathcal{Z}} X(z)$, ROC R

Then $x^*[n] \xleftrightarrow{\mathcal{Z}} X^*(z^*)$, ROC R

Notes: 1. if $x[n]$ is real-valued, then $X(e^{j\omega})$ is conjugate symmetric.

2. $\mathcal{E}v\{x[n]\} \xleftrightarrow{\mathcal{F}} \mathrm{Re}[X(e^{j\omega})]$

3. $\mathcal{O}d\{x[n]\} \xleftrightarrow{\mathcal{F}} \mathrm{Im}[X(e^{j\omega})]$

Time Reversal

if $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$

Then $x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$

if $x[n] \xleftrightarrow{\mathcal{Z}} X(z)$, ROC R

Then $x[-n] \xleftrightarrow{\mathcal{Z}} X(\frac{1}{z})$, ROC $\frac{1}{R}$

Time Expansion

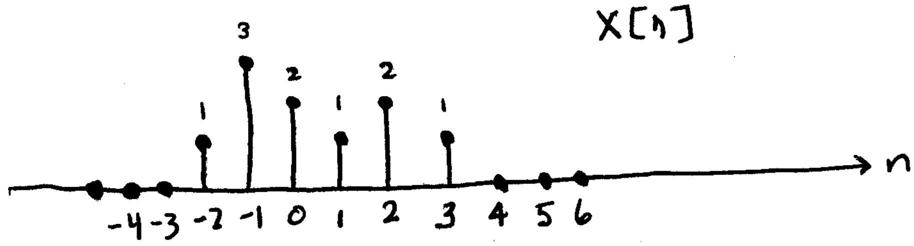
- In discrete time, it doesn't make sense to talk about the signal $x[an]$ when "an" is not an integer.
- But there is a related discrete-time property called "time expansion", or "upsampling".
- For a signal $x[n]$, consider the upsampled signal

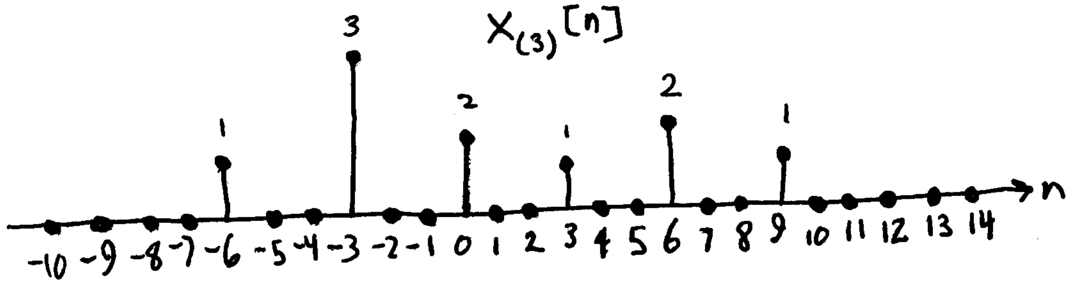
$$x_{(k)}[n] = \begin{cases} x[\frac{n}{k}], & \frac{n}{k} \in \mathbb{Z} \\ 0, & \frac{n}{k} \notin \mathbb{Z}. \end{cases}$$

where $k \in \mathbb{Z}$.

- Forming $x_{(k)}[n]$ is equivalent to inserting $k-1$ zeros between each sample of $x[n]$.

EX:





if $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$

then $x_{(k)}[n] \xleftrightarrow{\mathcal{F}} X(e^{jk\omega})$

if $x[n] \xleftrightarrow{\mathcal{Z}} X(z)$, ROC R

then $x_{(k)}[n] \xleftrightarrow{\mathcal{Z}} X(z^k)$, ROC $R^{\frac{1}{k}}$

Convolution

if $x_1[n] \xleftrightarrow{\mathcal{F}} X_1(e^{j\omega})$

and $x_2[n] \xleftrightarrow{\mathcal{F}} X_2(e^{j\omega})$

Then $x_1[n] * x_2[n]$

$$\xleftrightarrow{\mathcal{F}} X_1(e^{j\omega})\,X_2(e^{j\omega})$$

if $x_1[n] \xleftrightarrow{\mathcal{Z}} X_1(z)$, ROC R_1

and $x_2[n] \xleftrightarrow{\mathcal{Z}} X_2(z)$, ROC R_2

Then $x_1[n] * x_2[n]$

$$\xleftrightarrow{\mathcal{Z}} X_1(z)\,X_2(z), \text{ ROC } R_1 \cap R_2 .$$

Note: The ROC may be bigger if some poles cancel.

Frequency Convolution

(Fourier Transform only)

if $x_1[n] \xleftrightarrow{\mathcal{F}} X_1(e^{j\omega})$

and $x_2[n] \xleftrightarrow{\mathcal{F}} X_2(e^{j\omega})$

Then $x_1[n]\,x_2[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})\,X_2(e^{j(\omega-\theta)})\,d\theta$

Frequency and Z-domain Differentiation

if $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$

Then $n\,x[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(e^{j\omega})$

if $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, ROC R

Then $n\,x[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}X(z)$, ROC R.

Time Differencing

if $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$

Then

$x[n]-x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} (1-e^{-j\omega})\,X(e^{j\omega})$

if $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, ROC R

Then

$x[n]-x[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} (1-z^{-1})\,X(z)$,

with ROC at least $R \cap \{|z|>0\}$

Time Accumulation

if $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$

then $\displaystyle\sum_{k=-\infty}^{n} x[k] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}}\,X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega-2\pi k)$

if $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, ROC R,

then $\displaystyle\sum_{k=-\infty}^{n} x[k] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}\,X(z)$ with ROC at least $R \cap \{|z|>1\}$

Initial Value Theorem

(Z-transform only)

If $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ and $x[n]=0 \;\; \forall\, n<0$,

then $x[0] = \lim_{z\to\infty} X(z)$.

Parseval's Formula

(Fourier Transform only)

if $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$ and $x[n]$ is aperiodic,

then $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

NOTE: if H is a discrete-time LSI system with unit pulse response $h[n]$, input $x[n]$, and output $y[n]$

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = x[n] * h[n],$$

Then $$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) \leftarrow \text{System frequency response.}$$

$$\frac{Y(z)}{X(z)} = H(z) \leftarrow \text{system transfer function.}$$

Constant Coefficient Linear Difference Equations

- Recall: in continuous-time, if the input $x(t)$ and output $y(t)$ of an LSI system are related by a constant coefficient linear differential equation,
  - → Then the system frequency response is rational in ω
  - → The system transfer function is rational in s.

- Similarly, if the input $x[n]$ and output $y[n]$ of an LSI discrete-time system are related by a constant coefficient linear difference equation

$$a_0 y[n] + a_1 y[n-1] + a_2 y[n-2] + \dots + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M],$$

→ or: $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{\ell=0}^{M} b_\ell x[n-\ell], \qquad (*)$$

⇒ Then the system frequency response is rational in $e^{-j\omega}$

→ and the system transfer function is rational in z^{-1}.

- Taking Fourier transforms on both sides of $(*)$, we obtain

$$\sum_{k=0}^{N} a_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{\ell=0}^{M} b_\ell e^{-j\omega \ell} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{\ell=0}^{M} b_\ell e^{-j\omega\ell}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

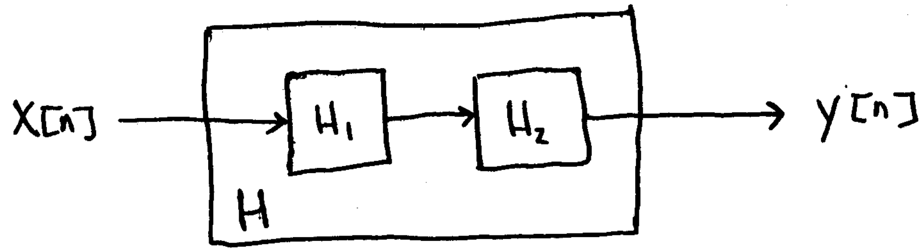
- Taking Z-transforms, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{\ell=0}^{M} b_\ell z^{-\ell} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Fourier Transform Analysis of Discrete-Time LSI Systems

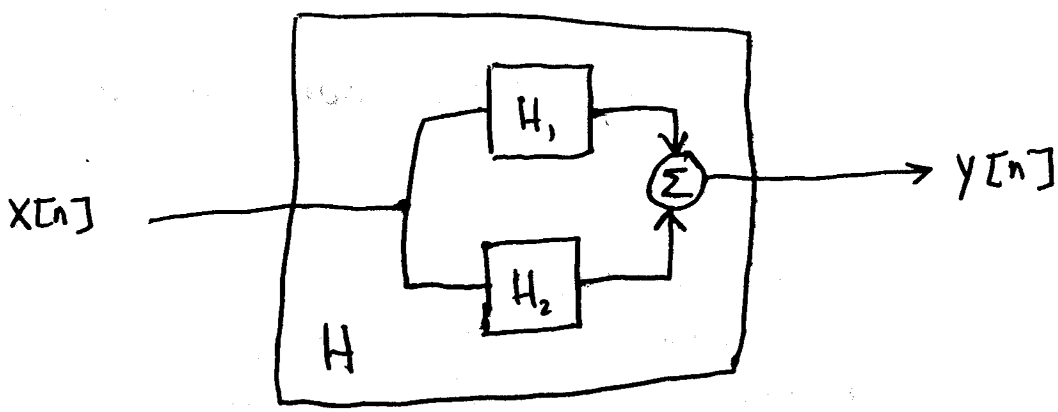
- Cascade connection:



$$h[n] = h_1[n] * h_2[n] = h_2[n] * h_1[n]$$

$$H(e^{j\omega}) = H_1(e^{j\omega})\, H_2(e^{j\omega})$$

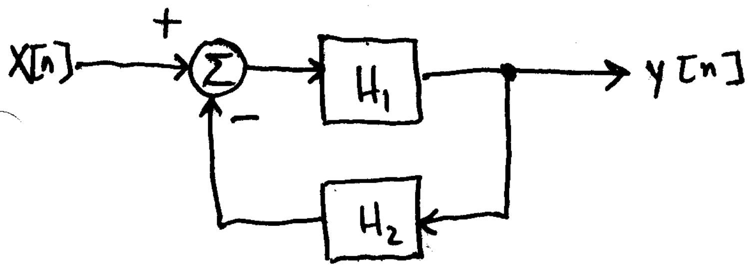
- Parallel Connection:



$$h[n] = h_1[n] + h_2[n]$$

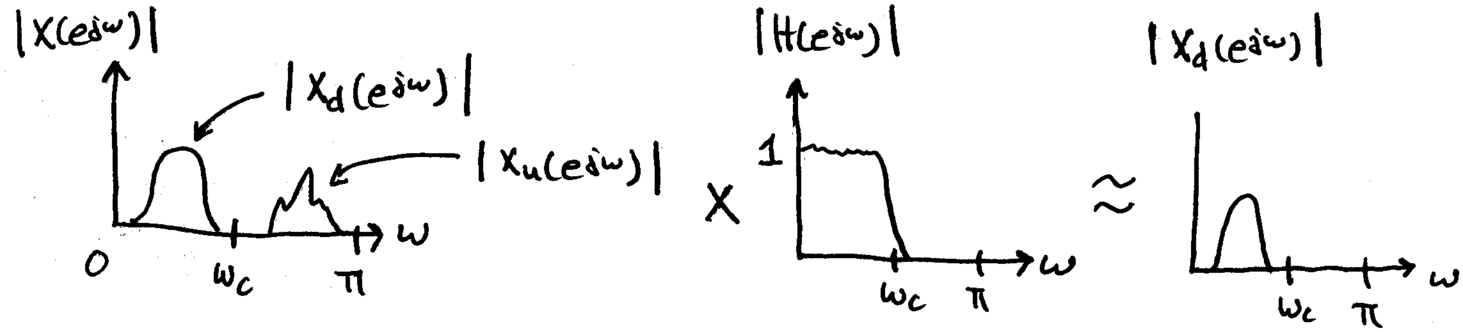
$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

- Feedback Connection:



$$H(e^{j\omega}) = \frac{H_1(e^{j\omega})}{1 + H_1(e^{j\omega})\, H_2(e^{j\omega})}$$

- Suppose $x[n]$ is the sum of a desirable part $x_d[n]$ and an undesirable part $x_u[n]$: $\quad x[n] = x_d[n] + x_u[n]$.
  - → $x_d[n]$ might be a digital telephone conversation and $x_u[n]$ might be noise.
- If $X_d(e^{j\omega})$ and $X_u(e^{j\omega})$ do not overlap, then we can design a filter frequency response $H(e^{j\omega})$ to remove the noise:

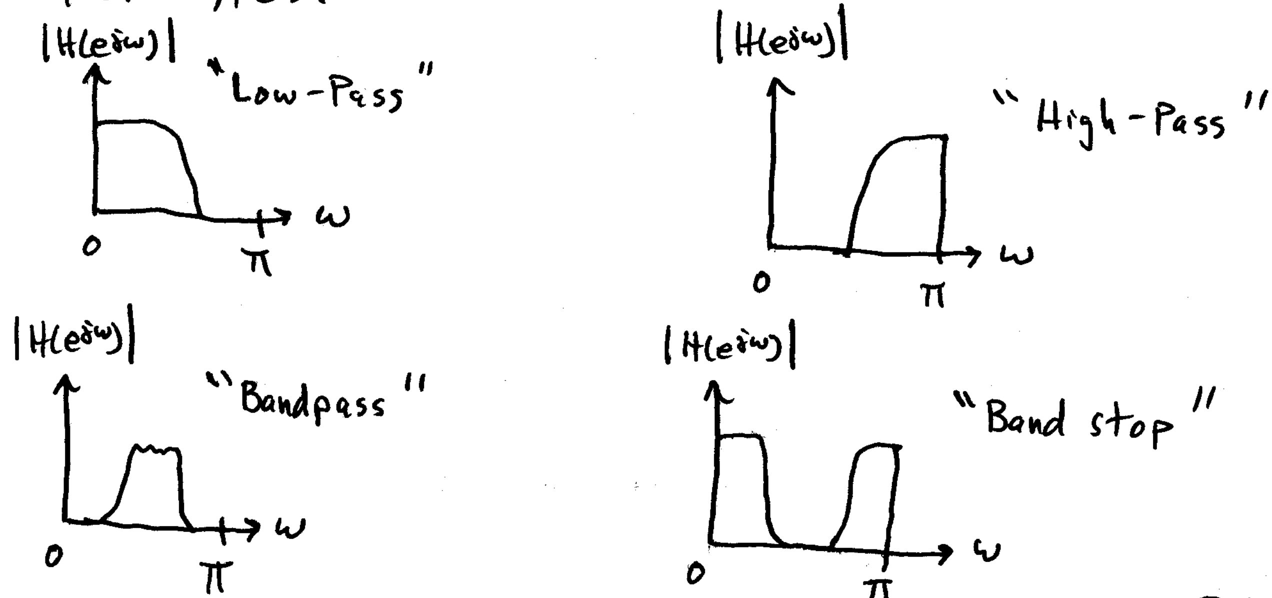


- The filter unit pulse response $h[n]$ is designed according to

$$h[n] = \mathcal{F}^{-1}\left[H(e^{j\omega})\right]$$

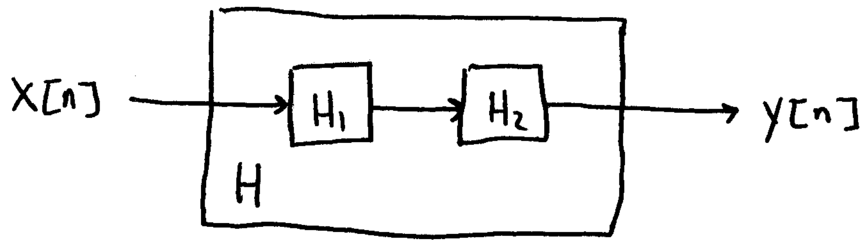
- Then $y[n] = x[n] * h[n] \approx x_d[n]$.

- Filter Types:



Z-domain LSI system Analysis

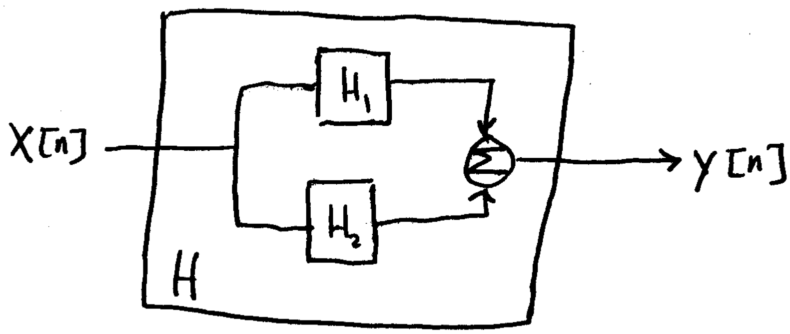
- **Cascade Connection:**



$$h[n] = h_1[n] * h_2[n]$$

$$H(z) = H_1(z)H_2(z)$$

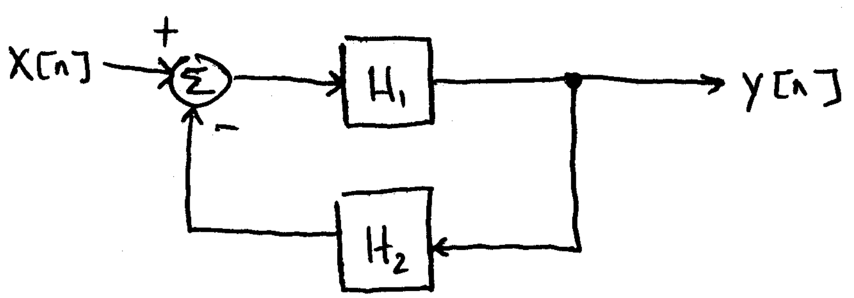
- **Parallel Connection:**



$$h[n] = h_1[n] + h_2[n]$$

$$H(z) = H_1(z) + H_2(z)$$

- **Feedback Connection:**



$$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

- if $h[n]$ is right-sided, then the ROC of $H(z)$ is the exterior of a circle in the z-plane.
- If $h[n]$ is causal, then $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$, which converges for $z = \infty$.

→

⇒ Putting these together, we have

FACT: A discrete-time LSI system H is causal if and only if the ROC of the transfer function is the exterior of a circle and contains the point $z=\infty$.

- if $h[n]$ is right-sided and $H(z)$ is rational, then the ROC of $H(z)$ is the exterior of a circle that contains all the finite poles of $H(z)$.

⇒ Adding this, we have

FACT: A discrete-time LSI system H with rational transfer function $H(z)$ is causal if and only if

1. The ROC of $H(z)$ is the exterior of a circle that contains all the poles of $H(z)$.
2. The numerator of $H(z)$ is of lower order than the denominator (no poles at $z=\infty$).

- If H is a BIBO stable discrete-time LSI system, then $\sum_{n=-\infty}^{\infty}|h[n]| < \infty$.

→ This means that the Fourier transform $H(e^{j\omega})$ exists.
→ This means that the unit circle is in the ROC of $H(z)$.

- we have

FACT: A discrete-time LSI system is BIBO stable if and only if the ROC of the transfer function contains the unit circle.

- Putting together everything from the last page, we get the "BIG" result:

<u>FACT</u>: A causal discrete-time LSI system is BIBO stable if and only if all of the poles of the transfer function lie <u>inside</u> the unit circle.

Unilateral Z-transform

- There is also a unilateral Z-transform defined by

$$X_u(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = Z_u\{x[n]\}$$

- It behaves pretty much analogously to the unilateral Laplace transform.

- There are two shifting properties

→ $Z_u\{x[n-1]\} = z^{-1} X_u(z) + x[-1]$

→ $Z_u\{x[n+1]\} = z X_u(z) - z x[0]$

- These properties make the unilateral Z-transform usefull for solving constant coefficient difference equations with initial conditions.

⇒ We are done with chapters 5 and 10.