ECE 4213/5213

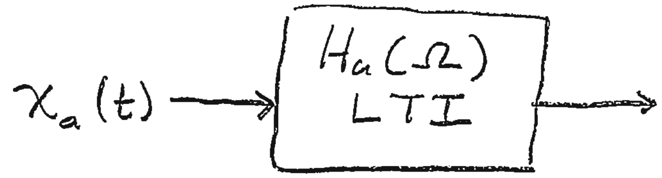
MODULE 3.8

Index

DIGITAL PROCESSING OF CONTINUOUS-TIME SIGNALS

- This topic is covered in sections 4.1-4.4 of the book.
- As we have been doing throughout this class, we will continue to use Ω for "continuous" frequency and ω for discrete or "digital" frequency:
  - $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega)$
  - $x[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\omega})$

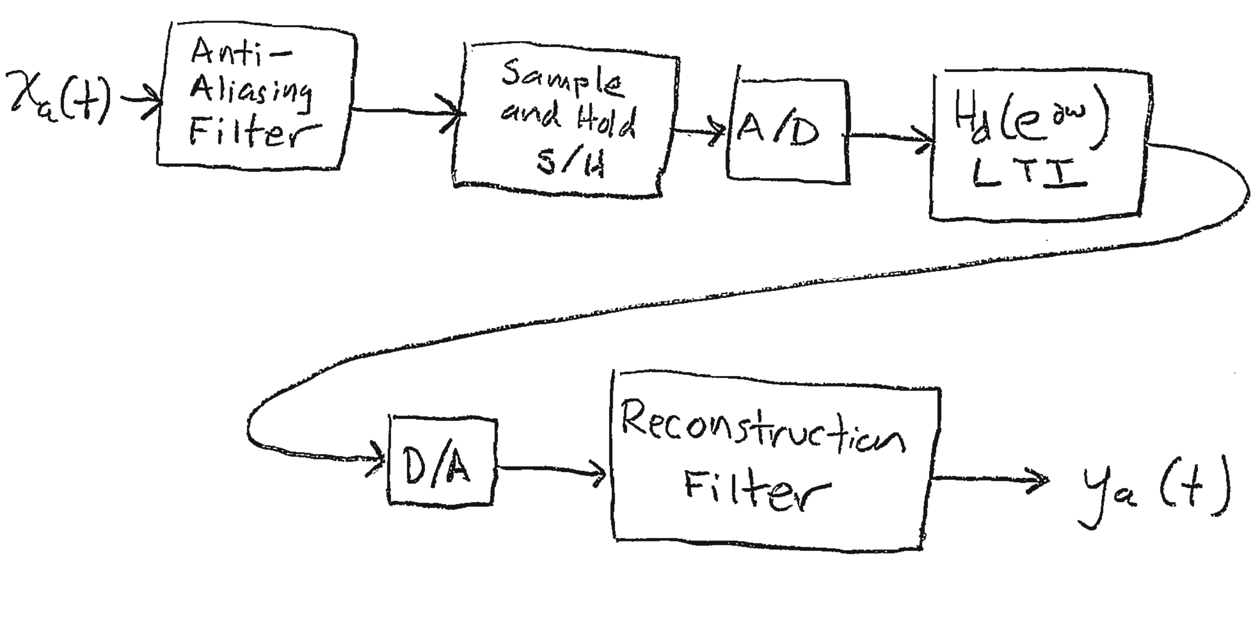
- The big idea is to implement an analog filter $H_a(\Omega)$ using an A/D converter, a digital filter $H_d(e^{j\omega})$, and a D/A converter.
- The analog filter to be implemented:



$$y_a(t) = x_a(t) * h_a(t)$$

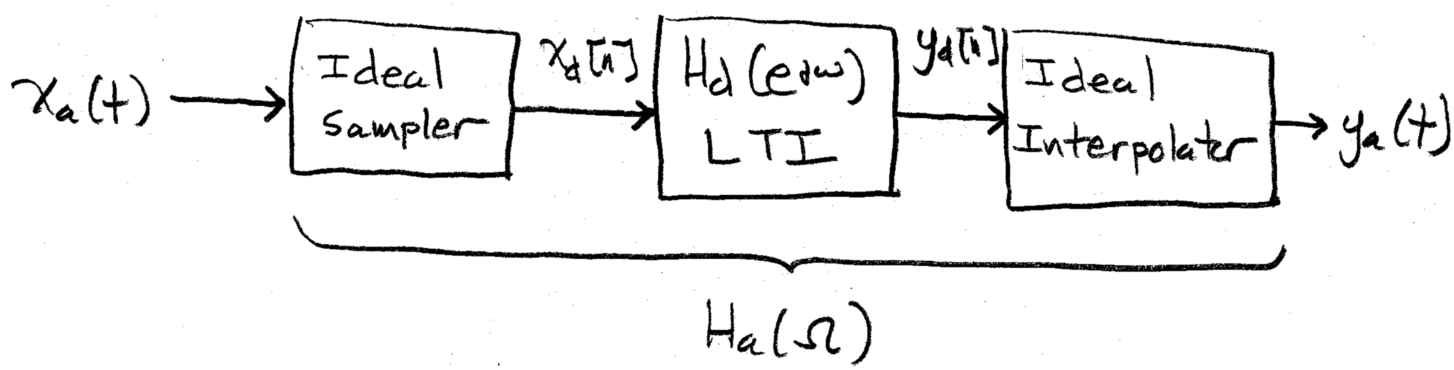
$$Y_a(\Omega) = X_a(\Omega) H_a(\Omega)$$

- In practice, it will be implemented like this:



- The anti-aliasing filter is an analog low-pass filter. It removes any high frequencies in $x_a(t)$ that would just be aliased anyway. It ensures that the analog input signal is band limited.
- The S/H circuit samples the analog input signal at discrete time instants and holds a constant value. It turns the analog input signal into a "stair step" analog signal that is piecewise constant.
- The A/D converter turns the "stair step" analog signal into a true digital signal that is discrete in both time and amplitude.
- $H_d(e^{j\omega})$ is a digital filter that is used to implement the analog frequency response $H_a(\Omega)$.
- The D/A converter turns the digital output signal into an analog signal that is discrete... it is an equi-spaced train of weighted impulses.
- The analog reconstruction filter interpolates to fill in the spaces between the impulses. It is a low-pass filter that creates the analog output signal.

- We will simplify the block diagram by:
  - ignoring the anti-aliasing filter,
  - combining the S/H circuit and A/D into an "Ideal Sampler", and
  - combining the D/A and reconstruction filter into an "Ideal Interpolator."
- So, our picture for analyzing this setup looks like:



- Why do this?
  1. The system is easy to change if the problem changes. Changing the filter will require new software, but not new hardware.
  2. One piece of hardware can be used to implement many different filters.
  3. In some cases, there may be large time delays between $x_a(t)$ and $y_a(t)$... for example, in CD audio applications, DVD and blu ray video, and magnetic disk storage systems.

- In such cases, the advantages of this setup include:
  - $x_d[n]$ does not degrade with time
  - $x_d[n]$ can be easily protected by error correcting codes (ex: parity).
  - copies of $x_d[n]$ and $y_d[n]$ are just as good as the original.
- We assume that the sampling is uniform in time, so that the ideal sampler takes one sample every T seconds. This gives us

$$x[n] = x_a(nT), \quad n \in \mathbb{Z}$$

- DEFINITIONS:

T : sampling interval (seconds)

$F_T = \frac{1}{T}$: sampling frequency (Hz)

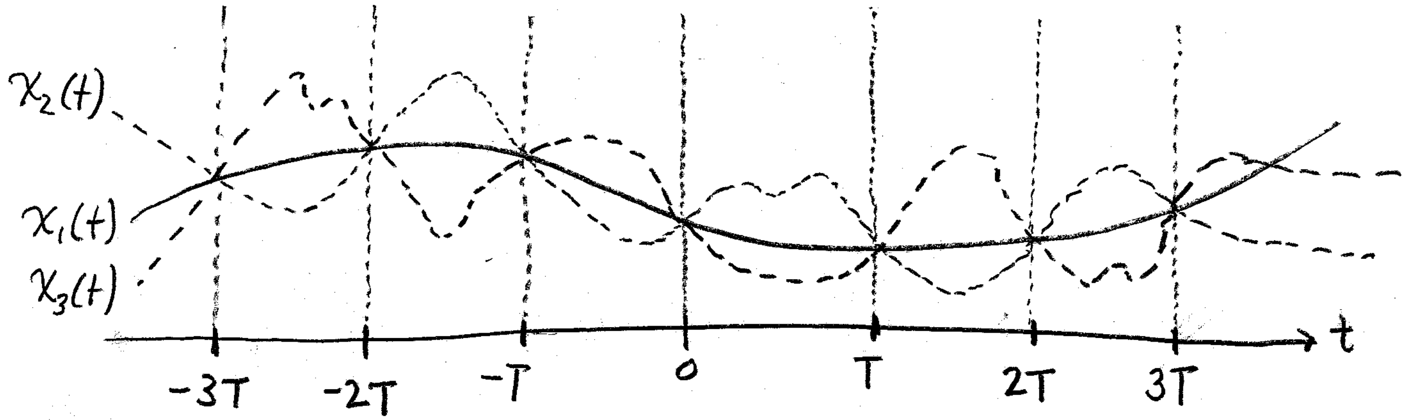
$\Omega_T = \frac{2\pi}{T} = 2\pi F_T$: Sampling frequency (rad/sec)

- To start with, we will let $H_d(e^{j\omega}) = 1$.
  - This makes $h_d[n] = \delta[n]$, the "do nothing" filter.
- Then we should get $y_a(t) = x_a(t)$.

Notation Differences Between Notes & Oppenheim/Schafer:

Notes	Book	Description
T	T	Sampling Interval/Period
Ω_T	Ω_s	Sampling Freq in Radians/Sec
F_T	f_S	Sampling Freq in Hz
$x_a(t) \leftrightarrow X_a(\Omega)$	$x_c(t) \leftrightarrow X_c(j\Omega)$	Analog/Continuous input signal
$y_a(t) \leftrightarrow Y_a(\Omega)$	$y_r(t) \leftrightarrow Y_r(j\Omega)$	Analog/Reconstructed output signal
Ω_M	Ω_N	Bandlimit of analog input signal
$x_d[n] \leftrightarrow X_d(e^{j\omega})$	$x[n] \leftrightarrow X(e^{j\omega})$	Digital input signal
$y_d[n] \leftrightarrow Y_d(e^{j\omega})$	$y[n] \leftrightarrow Y(e^{j\omega})$	Digital output signal
$s_a(t) \leftrightarrow S_a(\Omega)$	*None*	Any general periodic analog signal
$p(t) \leftrightarrow P(\Omega)$	$s(t) \leftrightarrow S(j\Omega)$	Periodic impulse train (for sampling)
$x_p(t) \leftrightarrow X_p(\Omega)$	$x_s(t) \leftrightarrow X_s(j\Omega)$	Analog input signal sampled by periodic impulse train
$h_R(t) \leftrightarrow H_R(\Omega)$	$h_r(t) \leftrightarrow H_r(j\Omega)$	Analog reconstruction filter
$h_d[n] \leftrightarrow H_d(e^{j\omega})$	$H(e^{j\omega})$	The digital filter (used to implement the analog filter)
$y_a(t) \leftrightarrow Y_a(\Omega)$	$x_r(t) \leftrightarrow X_r(j\Omega)$	Analog output signal when the digital filter is the "do nothing" filter
$H_a(\Omega)$	$H_{\mathrm{eff}}(j\Omega)$	The implemented analog filter

- Consider three analog signals:



→ We have $x_1(nT) = x_2(nT) = x_3(nT)$!!

- They all have the exact <u>same</u> samples!!
- So clearly some restrictions are needed!

- More definitions:

$$x_a(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X_a(\Omega)e^{j\Omega t}\,d\Omega \overset{\mathcal{F}}{\longleftrightarrow} X_a(\Omega) = \int_{\mathbb{R}} x_a(t)e^{-j\Omega t}\,dt$$

$$x_d[n] = \frac{1}{2\pi}\int_{2\pi} X_d(e^{j\omega})e^{j\omega n}\,d\omega \overset{\text{DTFT}}{\longleftrightarrow} X_d(e^{j\omega}) = \sum_{n\in\mathbb{Z}} x_d[n]e^{-j\omega n}$$

- To understand all of this, we will need to use some Fourier Series.

- Review: let $s_a(t)$ be periodic with period T.
  - Then $s_a(t)$ can be written in a Fourier series as:

$$s_a(t) = \sum_{k \in \mathbb{Z}} a_k e^{jk\Omega_T t}, \quad \text{where } \Omega_T = \frac{2\pi}{T}.$$

  - The Fourier series coefficients are given by

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} s_a(t) e^{-jk\Omega_T t}\, dt$$

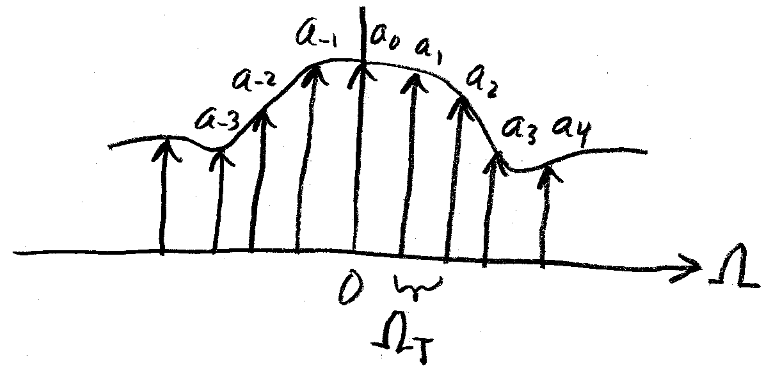
→ notice that this looks like a Fourier transform computed over just one period of $s_a(t)$.

- Now, by writing $s_a(t)$ in a Fourier series, we can compute the Fourier transform $S_a(\Omega)$ easily:

$$S_a(\Omega) = \mathcal{F}\{s_a(t)\} = \mathcal{F}\left\{\sum_{k \in \mathbb{Z}} a_k e^{jk\Omega_T t}\right\}$$

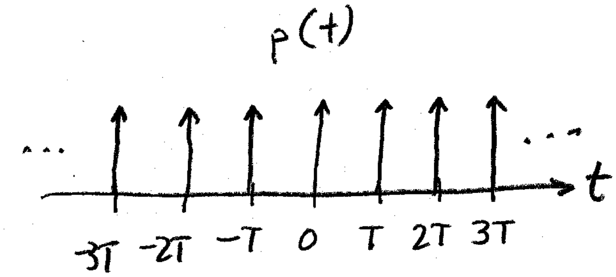
$$= \sum_{k \in \mathbb{Z}} a_k \mathcal{F}\{e^{jk\Omega_T t}\} = 2\pi \sum_{k \in \mathbb{Z}} a_k \delta(\Omega - k\Omega_T)$$

- This shows that $S_a(\Omega)$ is a weighted train of equidistant impulses.



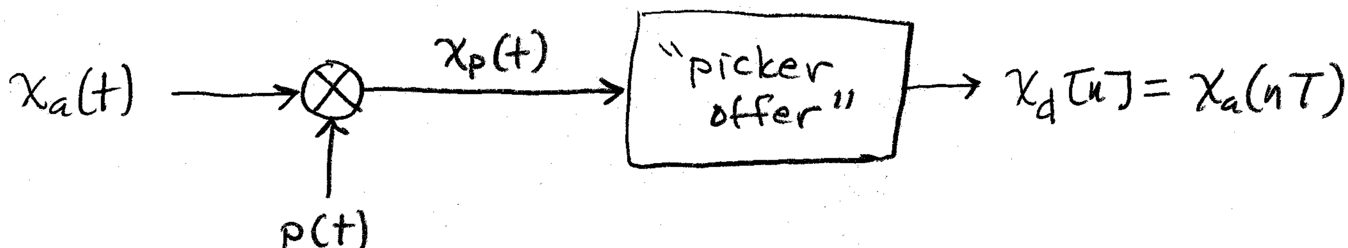
- Now consider the analog signal

$$p(t) = \sum_{n \in \mathbb{Z}} \delta(t - nT)$$



- Note that $p(t)$ is periodic with period T.

- Our model for the ideal sampler on page 3.8-3:



- To understand what happens when $H_d(e^{j\omega})$ is the "do nothing" filter, we need to know $X_p(\Omega)$.
- To find it, we first write $p(t)$ in a Fourier series:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{-jk\Omega_T t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \left[\sum_{n \in \mathbb{Z}} \delta(t - nT)\right] e^{-jk\Omega_T t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\Omega_T t}\, dt$$

{ since only the $n=0$ term of the sum is "turned on" within the limits of integration

$$= \frac{1}{T} \cdot 1 = \frac{1}{T}.$$

So $a_k = \frac{1}{T} \quad \forall k$.

- Writing $p(t)$ in a Fourier series, we have

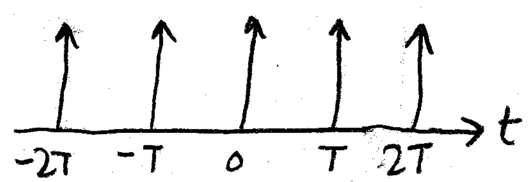
$$p(t) = \sum_{k \in \mathbb{Z}} a_k e^{jk\Omega_T t} = \frac{1}{T} \sum_{k \in \mathbb{Z}} e^{jk\Omega_T t}$$

- Now, taking the Fourier transform, we get

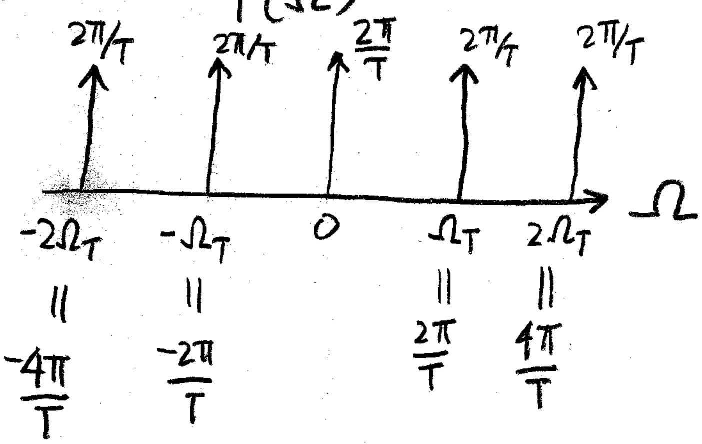
$$P(\Omega) = \mathcal{F}\{p(t)\} = \frac{1}{T} \sum_{k \in \mathbb{Z}} \mathcal{F}\{e^{jk\Omega_T t}\}$$

$$= \frac{2\pi}{T} \sum_{k \in \mathbb{Z}} \delta(\Omega - k\Omega_T)$$

$p(t)$



$P(\Omega)$



- Notice the "reciprocal spreading":

  - if the pulses in $p(t)$ are close together (small T, fast sampling), then the pulses in $P(\Omega)$ are far apart (big Ω_T).

  - and vice-versa.

- Going back to our model of the ideal sampler on page 3.8-7, we need to know $X_p(\Omega)$.

$$x_p(t) = x_a(t)\,p(t)$$

- So,

$$X_p(\Omega) = \frac{1}{2\pi} X_a(\Omega) * P(\Omega)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} X_a(\Omega-\theta)\,P(\theta)\,d\theta$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} X_a(\Omega-\theta)\,\frac{2\pi}{T} \sum_{k\in\mathbb{Z}} \delta(\theta - k\Omega_T)\,d\theta$$

$$= \frac{1}{T} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} X_a(\Omega-\theta)\,\delta(\theta-k\Omega_T)\,d\theta$$

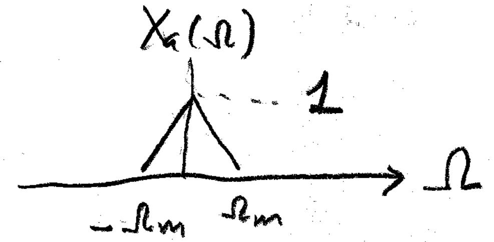
$$= \frac{1}{T} \sum_{k\in\mathbb{Z}} X_a(\Omega - k\Omega_T)$$

$\Rightarrow$ - The $k=0$ term gives us a copy of $X_a(\Omega)$

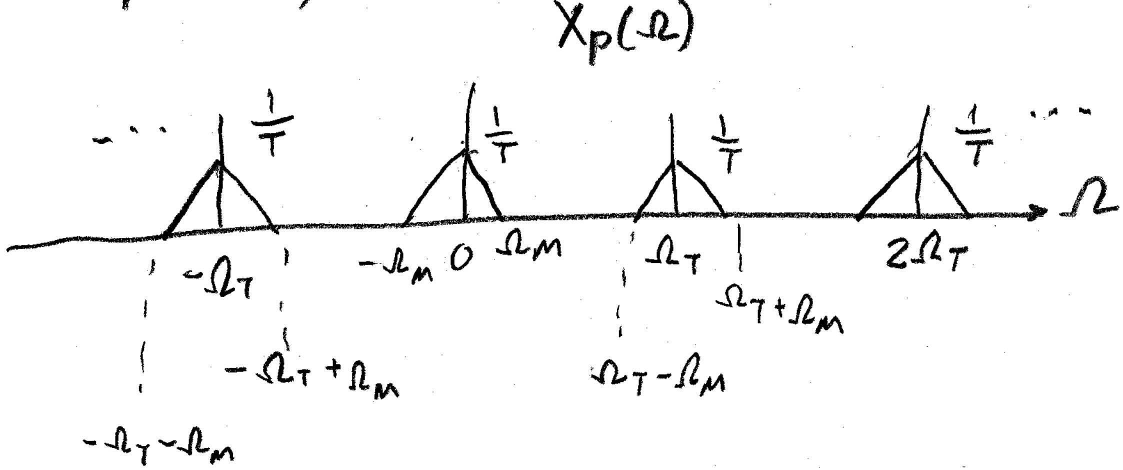
- The $k=1$ term gives us another copy centered at Ω_T.
- The $k=2$ term gives us another copy at $2\Omega_T$
- The $k=-1$ term gives us another copy at $-\Omega_T$
- The $k=-2$ term gives us another copy at $-2\Omega_T$

... and so on ... $\rightarrow$ all scaled by $\frac{1}{T}$

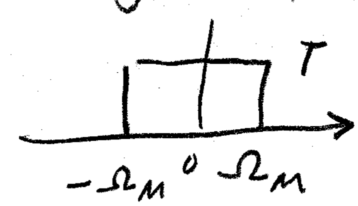
EX: Let $x_a(t)$ be bandlimited so that $X_a(\Omega) = 0$ for $|\Omega| > \Omega_m$:



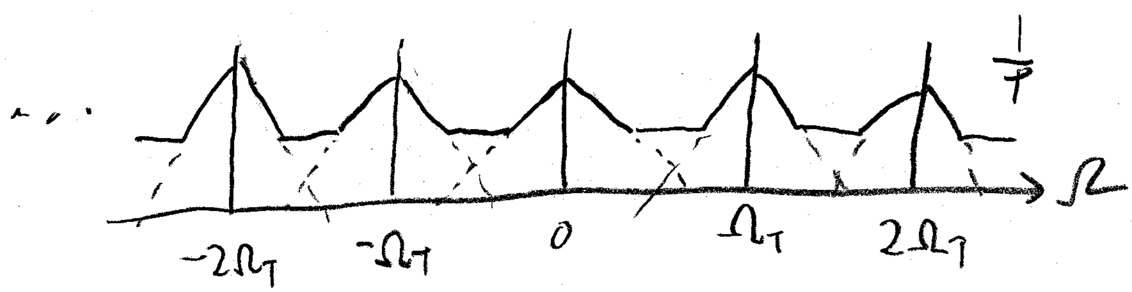
→ If you sample fast, the spectral replicas don't overlap:



→ you can recover $x_a(t)$ from $X_p(\Omega)$ using a low pass filter with a passband gain of T:



→ If you sample too slowly, the periodic repetitions overlap and you cannot recover $x_a(t)$: aliasing



- To prevent aliasing, we need

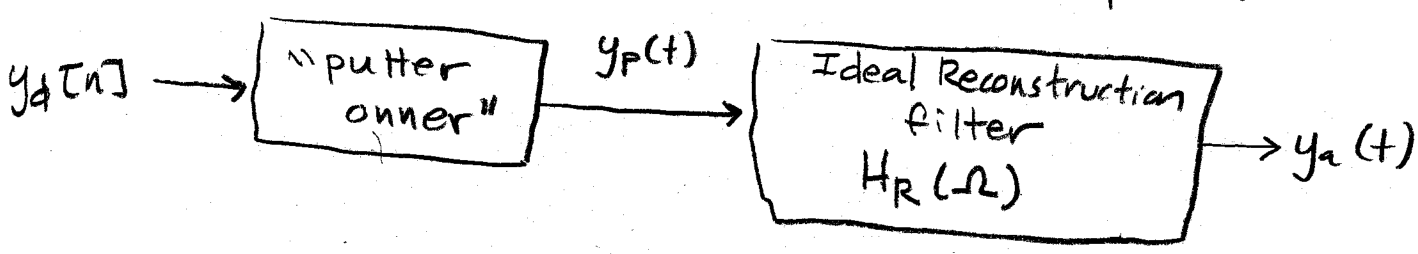
$$\Omega_M < \Omega_T - \Omega_M \quad \text{so the copies don't overlap}$$

$$2\Omega_M < \Omega_T \longrightarrow \Omega_T > 2\Omega_M$$

$$\text{or:} \quad T < \frac{\pi}{\Omega_M}$$

- This is the famous "Shannon sampling theorem!"
- The minimum sampling rate to prevent aliasing, so that $x_a(t)$ can be recovered from the samples, is $2\Omega_M$. This is called the "Nyquist rate."

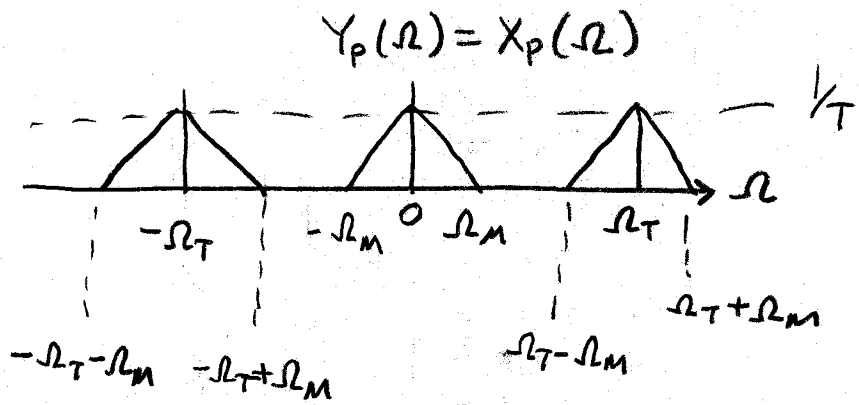
- Now, here's our model for the ideal interpolator:



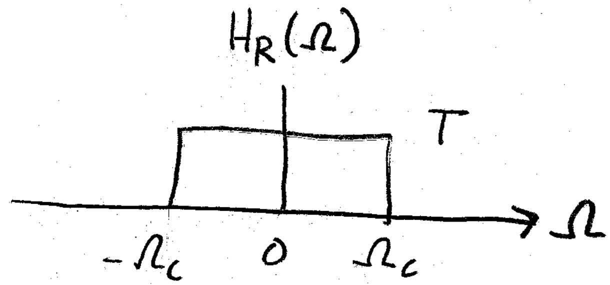
- To see what we need for $H_R(\Omega)$, observe that when $H_d(e^{j\omega}) = 1$ ("do nothing filter"), we have $y_d[n] = x_d[n]$ and so $y_p(t) = x_p(t)$.

- We need to design $H_R(\Omega)$ so that $y_a(t) = x_a(t)$.

- In the frequency domain, we have



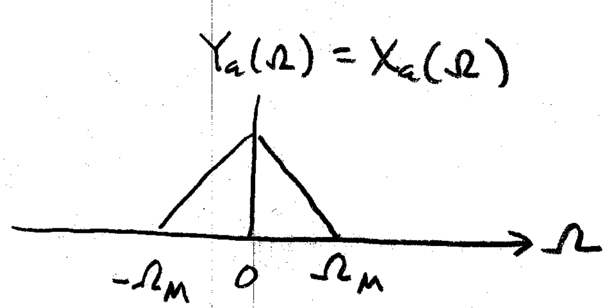
- To get back $x_a(t)$, we need:



with:

$$\Omega_M < \Omega_c < \Omega_T - \Omega_M$$

- This will give us



- So $H_R(\Omega)$ needs to be an ideal low-pass filter with a passband gain of T and a cutoff frequency Ω_c that lies between Ω_M and $\Omega_T - \Omega_M$.

- The impulse response is given by

$$h_R(t) = \frac{T \sin \Omega_c t}{\pi t}$$

- not causal
- not stable

→ In practice, this must be approximated by a practical analog low-pass filter that is causal and stable.

- For the ideal reconstruction filter, we have

$$y_a(t) = y_p(t) * h_R(t)$$

$$= \left[\sum_{n=-\infty}^{\infty} y_a(nT)\,\delta(t-nT)\right] * h_R(t)$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y_a(nT)\,\delta(\tau-nT)\,h_R(t-\tau)\,d\tau$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left[y_a(nT)\,h_R(t-\tau)\right]\delta(\tau-nT)\,d\tau$$

$$= \sum_{n=-\infty}^{\infty} y_a(nT)\,h_R(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} y_a(nT)\,\frac{T \sin[\Omega_c(t-nT)]}{\pi(t-nT)}$$

- This is called "Shannon reconstruction"
- It shows that each point in the reconstructed signal is a sync-weighted interpolation of all the samples.

- It may seem strange to you that different points in the signal $y_a(t)$ are not independent.

→ Since we currently have $H_d(e^{j\omega}) = 1$ ("do nothing filter") and $y_a(t) = x_a(t)$,

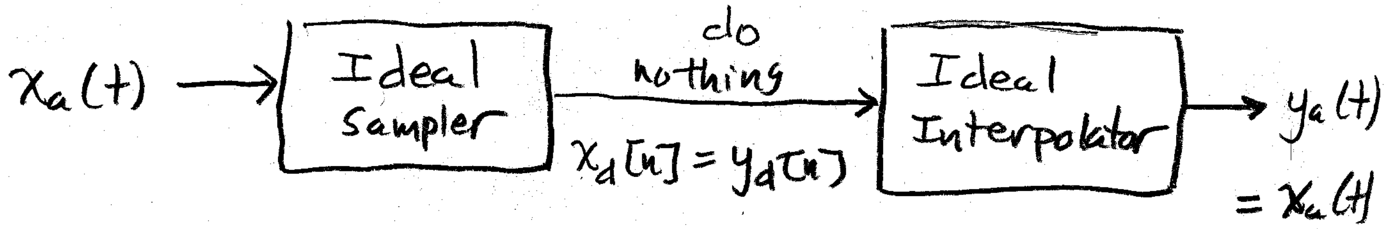
- this also means that different points in $x_a(t)$ are also not independent.

→ But remember that we are assuming that $x_a(t)$ is band limited (and thus so is $y_a(t)$).

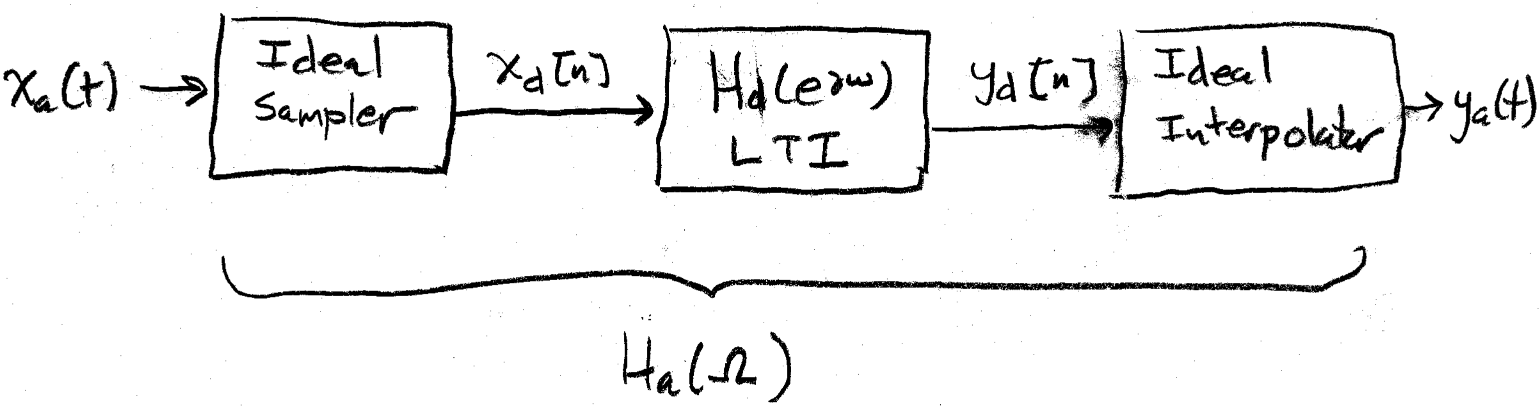
→ So once you fix the value of the signal at one point in time, it constrains the value at the rest of the times... totally arbitrary changes between two times are not possible because the frequency content of the signal is limited.

NOTE: I will sometimes refer to the ideal sampler as an "ideal A/D" and the ideal interpolator as an "ideal D/A."

→ We now change from the "do nothing" filter:



To a "do something" filter:



– We have:

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\,\delta(t-nT)$$

so

$$X_p(\Omega) = \mathcal{F}\{x_p(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x_a(nT)\,\delta(t-nT)\right\}$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\,\mathcal{F}\{\delta(t-nT)\}$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\,e^{-j\Omega nT} \qquad (*)$$

- Also, by definition,

$$X_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_d[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\, e^{-j\omega n} \qquad (**)$$

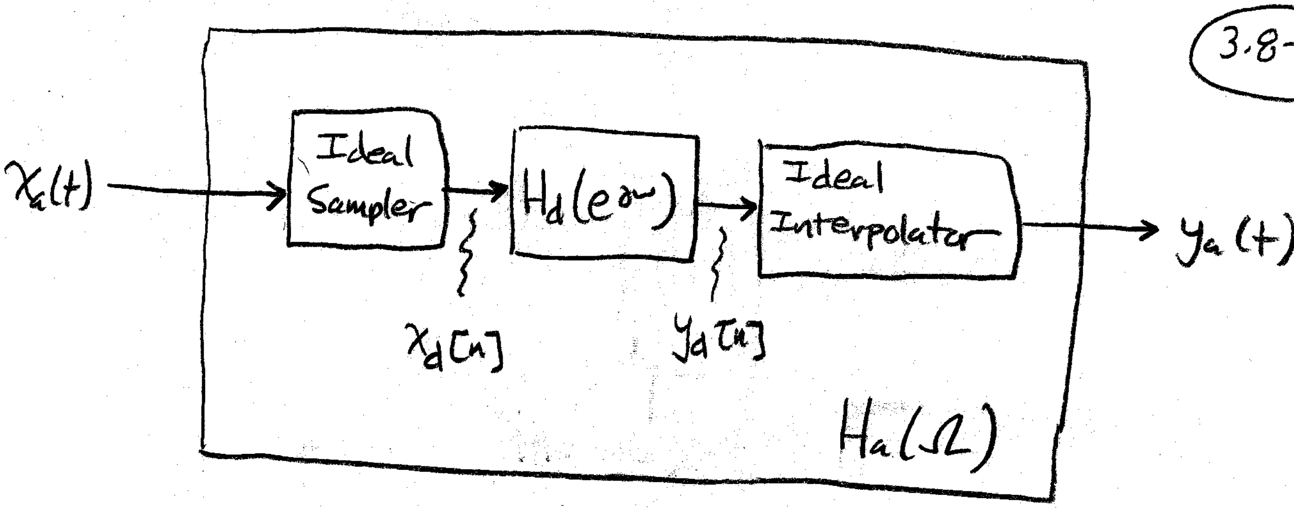
- Comparing (**) above to (*) on page 3.8-15 (They are equal!!), we have $\omega = \Omega T$, $\Omega = \frac{\omega}{T}$, and

$$X_d(e^{j\omega}) = X_p(\Omega)\Big|_{\Omega = \frac{\omega}{T}} = X_p\left(\frac{\omega}{T}\right)$$

- Ideal Sampler relations:

$$\boxed{\begin{aligned} \Omega &= \frac{\omega}{T} \\ \omega &= \Omega T \\ X_d(e^{j\omega}) &= X_p\left(\frac{\omega}{T}\right) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega}{T} - k\Omega_T\right) \end{aligned}}$$

$$x_a(t) \longrightarrow \boxed{\text{Ideal Sampler}} \longrightarrow x_d[n]$$



- At the terminals of the digital filter, we have

$$Y_d(e^{j\omega}) = X_d(e^{j\omega})\, H_d(e^{j\omega})$$

- Using $\omega = \Omega T$ to write this in terms of Ω, we have

$$Y_d(e^{j\Omega T}) = X_d(e^{j\Omega T})\, H_d(e^{j\Omega T})$$

- After the "putter onner" that converts $y_d[n]$ to $y_p(t)$, we have

$$Y_p(\Omega) = X_p(\Omega)\, H_d(e^{j\Omega T})$$

- After the reconstruction filter, this becomes

$$Y_a(\Omega) = X_a(\Omega)\, \underbrace{H_d(e^{j\Omega T})}_{\text{only the fundamental period is retained from } -\frac{\Omega_T}{2} \text{ to } +\frac{\Omega_T}{2}}$$

- Comparing the last equation on page 3.8-17 to $Y_a(\Omega) = X_a(\Omega) H_a(\Omega)$,

we see that

$$H_a(\Omega) = H_d(e^{j\Omega T}) \quad \left(\begin{array}{c}\text{fundamental}\\ \text{period only}\end{array}\right)$$

$$H_a(\Omega) = \begin{cases} H_d(e^{j\Omega T}), & |\Omega| \leq \frac{\Omega_T}{2} \left(= \frac{\pi}{T}\right) \\ 0, & |\Omega| > \frac{\Omega_T}{2} \left(= \frac{\pi}{T}\right) \end{cases} \quad ☆$$

→ This is the formula to obtain $H_a(\Omega)$ from $H_d(e^{j\omega})$.

- To obtain $H_d(e^{j\omega})$ from $H_a(\Omega)$:

$$H_d(e^{j\omega}) = H_a\left(\frac{\omega}{T}\right), \quad |\omega| < \pi \quad ☆$$

→ and periodically extend for $|\omega| \geq \pi$.

<u>NOTE:</u> Even though multiplication by $p(t)$ is not a linear operation (A/D and D/A converters are not linear),

→ The <u>overall</u> <u>system</u> $H_a(\Omega)$ <u>is</u> LTI if the sampling rate is at or above the Nyquist rate.

EX: Digital Differentiator

Recall: Fourier transform time differentiation property:

if $x_a(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_a(\Omega)$,

then $\frac{d}{dt} x_a(t) \stackrel{\mathcal{F}}{\longleftrightarrow} j\Omega X_a(\Omega)$

$\Rightarrow$ So a continuous-time system with frequency response $H_a(\Omega) = j\Omega$ is a differentiator:

$$x_a(t) \longrightarrow \boxed{H_a(\Omega) = j\Omega} \longrightarrow y_a(t) = \frac{d}{dt} x_a(t)$$

- Suppose the input signals are band limited to $|\Omega| \leq \Omega_M$, so that $X_a(\Omega) = 0 \ \forall \ |\Omega| > \Omega_M$.
- For digital implementation, we must sample at $\Omega_T > 2\Omega_M$.
- Since $\Omega_T = \frac{2\pi}{T}$, this means that

$$\frac{2\pi}{T} > 2\Omega_M$$

$$\Omega_M < \frac{\pi}{T}$$

- So we can implement a band limited differentiator with

$$H_a(\Omega) = \begin{cases} j\Omega, & |\Omega| \leq \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$

- For $|\omega| \leq \pi$, one period of the discrete-time frequency response is therefore given by

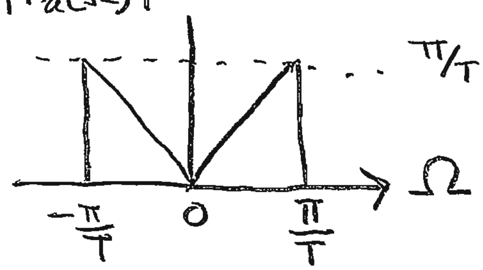
$$H_d(e^{j\omega}) = \begin{cases} j\frac{\omega}{T}, & \left|\frac{\omega}{T}\right| \leq \frac{\pi}{T} \\ 0, & \left|\frac{\omega}{T}\right| > \frac{\pi}{T} \end{cases}$$

$$= \begin{cases} j\frac{\omega}{T}, & |\omega| \leq \pi \\ 0, & |\omega| > \pi \end{cases}$$

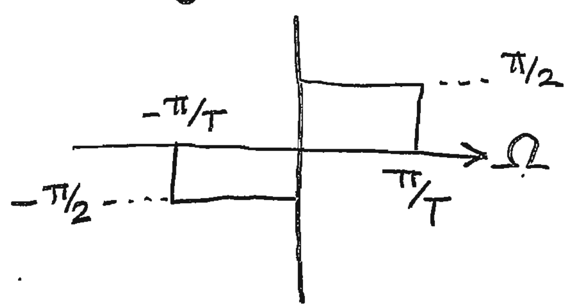
→ And then periodically extend this for $|\omega| > \pi$.

PICTURES

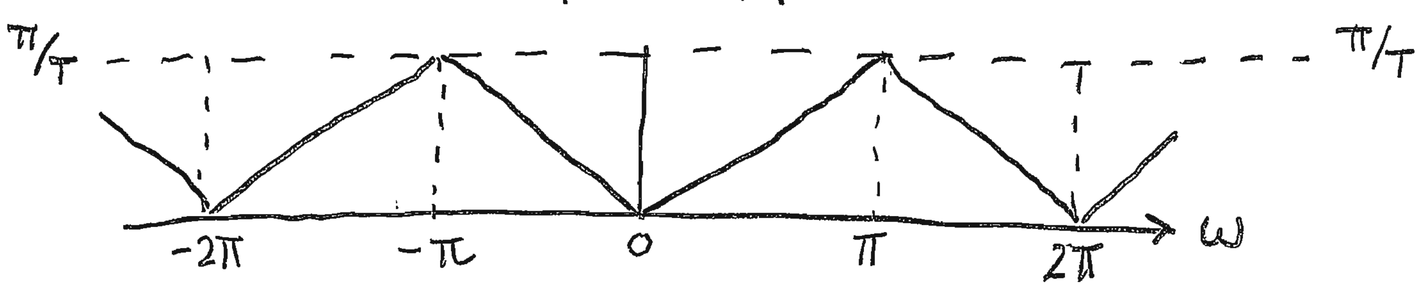
$|H_a(\Omega)|$



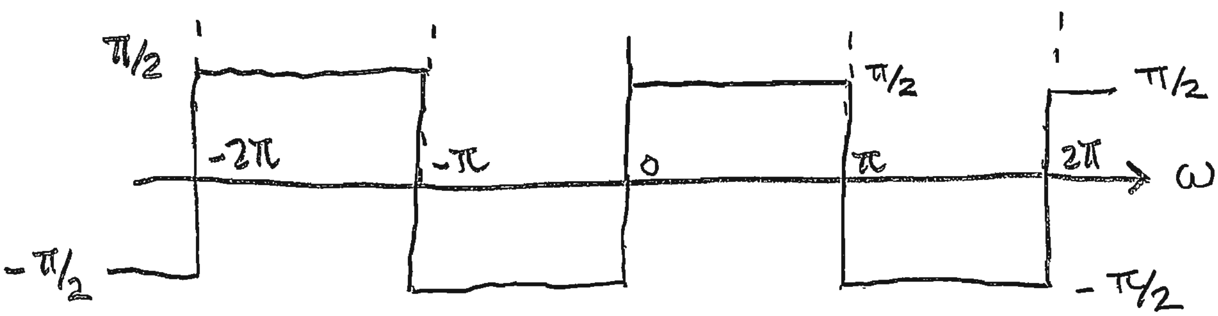
$\arg H_a(\Omega)$



$|H_d(e^{j\omega})|$



$\arg H_d(e^{j\omega})$



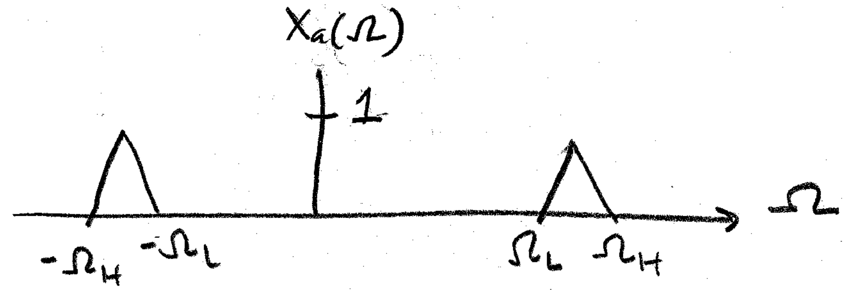
Bandpass Sampling:

3.8-21

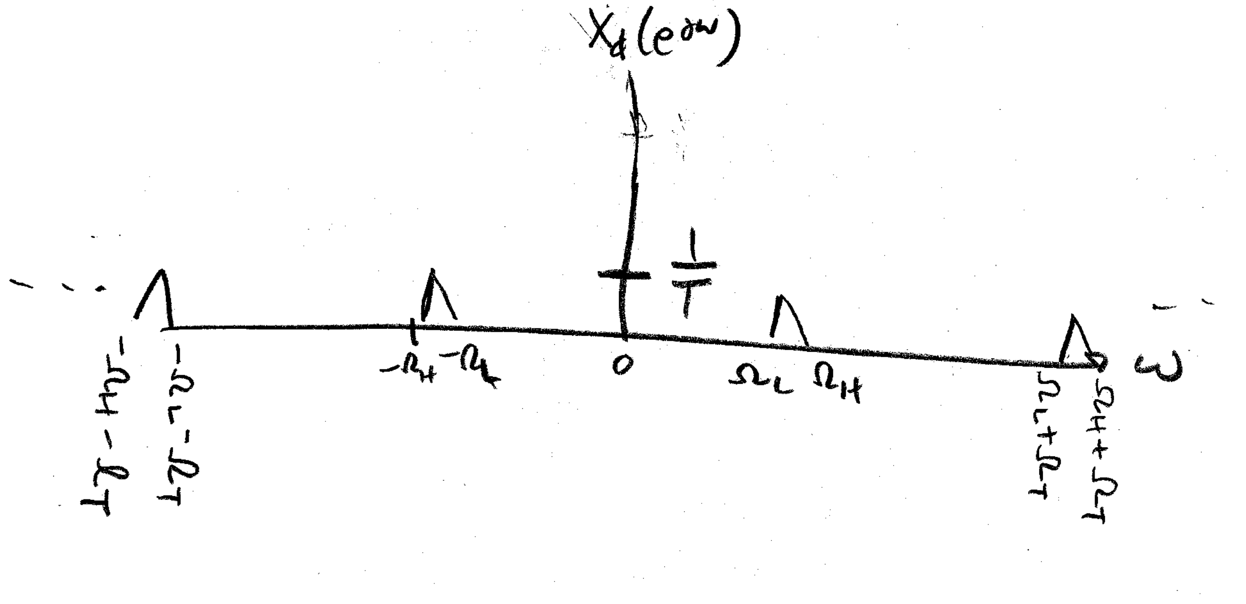
- Suppose that $x_a(t)$ is a real-valued band pass signal,

  such that $X_a(\Omega)$ is nonzero only for

  $-\Omega_H < \Omega < -\Omega_L$ and $\Omega_L < \Omega < \Omega_H$:



- The Nyquist rate for this signal is $2\Omega_H$. If we sample with $\Omega_T > 2\Omega_H$, then there will be no aliasing.
  $\Rightarrow$ but the digital spectrum will be inefficiently used in the sense that $X_d(e^{j\omega})$ will be zero over large portions of the frequency domain:



"Bandpass sampling" is a technique for sampling $x_a(t)$ with a sampling frequency that is <u>lower</u> than the Nyquist rate $2\Omega_H$, while still ensuring that aliasing will not occur. (3.8-22)

- Let $\Delta\Omega = \Omega_H - \Omega_L$, the "bandwidth" of the signal $x_a(t)$.

- Choose $\hat{\Omega}$ so that $\hat{\Omega} > \Omega_H$ and $\hat{\Omega} = M(\Delta\Omega)$ for some integer M.

- Then sample $x_a(t)$ with a sampling frequency

$$\Omega_T = 2(\Delta\Omega) = \frac{2\hat{\Omega}}{M}.$$

⇒ As long as there was lots of "empty space" in the original spectrum $X_a(\Omega)$, this will result in large M, so

$$\Omega_T = \frac{2\hat{\Omega}}{M} \approx \frac{2\Omega_H}{M} << 2\Omega_H$$

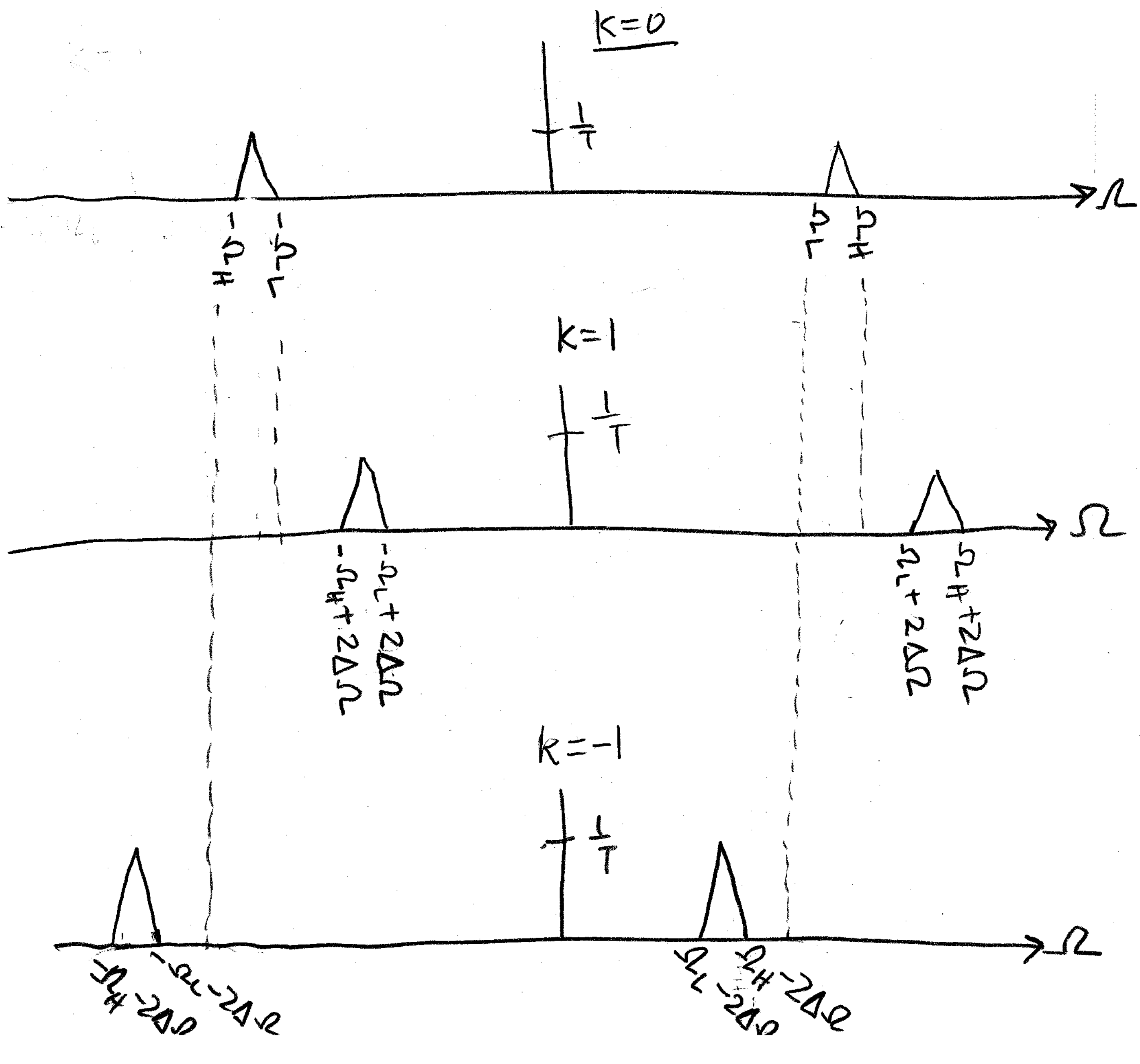
The Nyquist rate.

⇒ e.g., we will be sampling at a frequency that is <u>well below</u> the Nyqust frequency.

- Now, it follows from the last expression on page 3.8-22 that the Fourier spectrum of the impulse sampled signal $x_p(t) = x_a(t)\,p(t)$ is given by

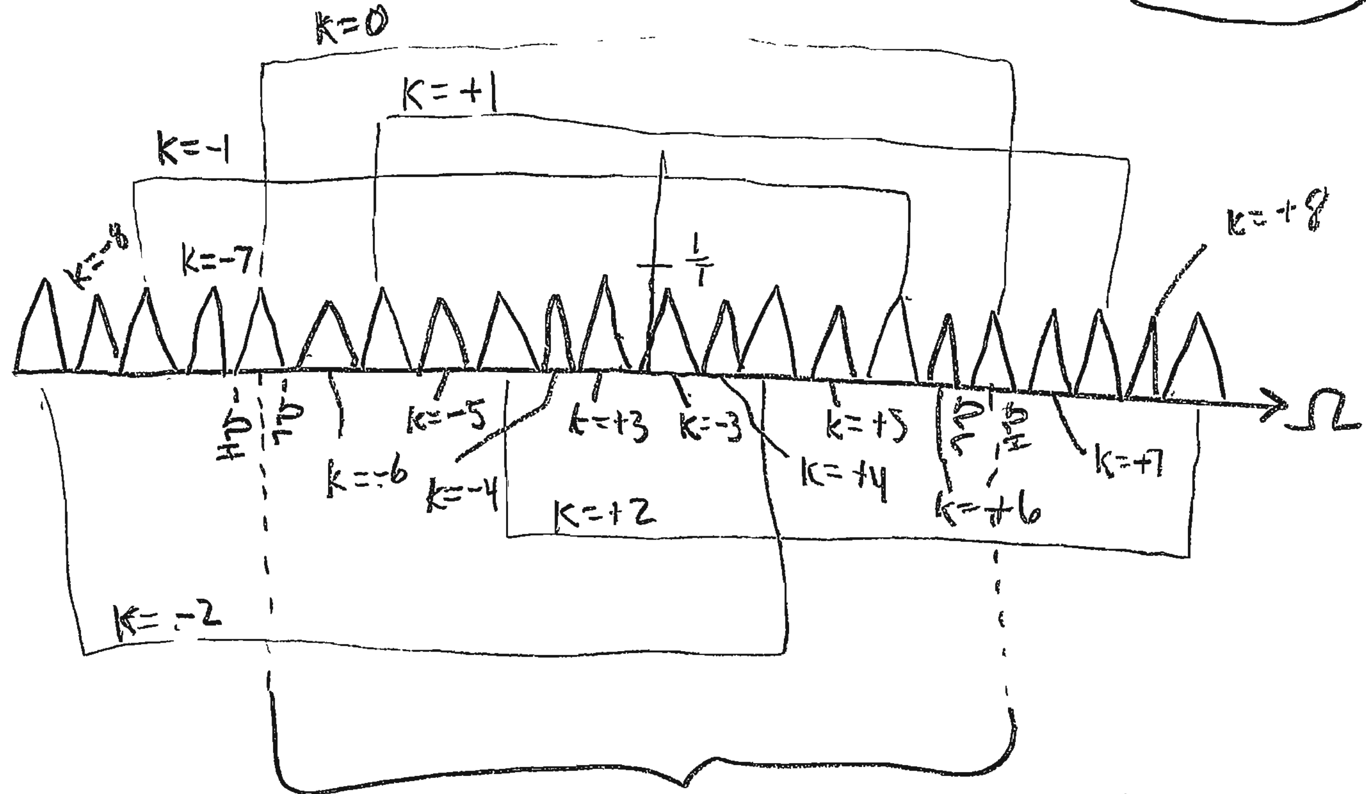
$$X_p(\Omega) = \frac{1}{T}\sum_{k\in\mathbb{Z}} X_a(\Omega - k\Omega_T)$$

$$= \frac{1}{T}\sum_{k\in\mathbb{Z}} X_a(\Omega - 2k(\Delta\Omega))$$



All Together:

3.8-24



There will be a total of $\underline{\underline{M}}$ shifted copies of the spectrum located within this range.

$\Rightarrow$ the original signal $x_a(t)$ can be recovered by using a bandpass reconstruction filter with gain T: