ECE 4213/5213
Test 2

Tuesday, November 25, 2025
4:30 PM - 5:45 PM

Fall 2025

Name: SOLUTION

Dr. Havlicek

Student Num:

Directions: This test is open book and open notes. You may also use your calculator and the course formula sheet. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ______________________ Date: ______________________

1. **25/20 pts**. Let $x[n]$ and $h[n]$ be finite-length discrete-time signals given by

$$\begin{aligned} x[n] &= [4 \;\; -2 \;\; -1 \;\;\; 1 \;\;\; 2] \\ &= 4\delta[n] - 2\delta[n-1] - \delta[n-2] + \delta[n-3] + 2\delta[n-4], \;\; 0 \le n \le 4, \end{aligned}$$

and

$$\begin{aligned} h[n] &= [-1 \;\;\; 5 \;\; -3 \;\; -1] \\ &= -\delta[n] + 5\delta[n-1] - 3\delta[n-2] - \delta[n-3], \;\; 0 \le n \le 3. \end{aligned}$$

Use the DFT to find the **circular** convolution $y[n] = x[n] \circledS h[n]$.

Length $\{x[n]\} = 5$. Length $\{h[n]\} = 4$. $\mapsto$ zero pad $h[n]$ to length 5 and use $W_5 = e^{-j2\pi/5}$. $\mapsto h'[n] = [-1 \;\; 5 \;\; -3 \;\; -1 \;\; 0]$.

$$X[k] = \sum_{n=0}^{3} x[n] W_5^{nk} = 4 - 2W_5^{k} - W_5^{2k} + W_5^{3k} + 2W_5^{4k} \quad , \; 0 \le k \le 4$$

$$H[k] = \sum_{n=0}^{3} h'[n] W_5^{nk} = -1 + 5W_5^{k} - 3W_5^{2k} - W_5^{3k} \quad , \; 0 \le k \le 4$$

$$\begin{aligned} Y[k] &= X[k]H[k] \\ &= -4 + 20W_5^{k} - 12W_5^{2k} - 4W_5^{3k} \\ &\quad + 2W_5^{k} - 10W_5^{2k} + 6W_5^{3k} + 2W_5^{4k} \\ &\quad + W_5^{2k} - 5W_5^{3k} + 3W_5^{4k} + W_5^{5k} \\ &\quad - W_5^{3k} + 5W_5^{4k} - 3W_5^{5k} - W_5^{6k} \\ &\quad - 2W_5^{4k} + 10W_5^{5k} - 6W_5^{6k} - 2W_5^{7k} \end{aligned}$$

$$Y[k] = -4 + 22W_5^{k} - 21W_5^{2k} - 4W_5^{3k} + 8W_5^{4k} + 8\underbrace{W_5^{5k}}_{1} - 7\underbrace{W_5^{6k}}_{W_5^{k}} - 2\underbrace{W_5^{7k}}_{W_5^{2k}}$$

$$\begin{aligned} &= 4 + 15W_5^{k} - 23W_5^{2k} - 4W_5^{3k} + 8W_5^{4k} \\ &= \sum_{n=0}^{4} y[n] W_5^{nk} \;\; // \end{aligned}$$

$$\boxed{\begin{aligned} y[n] &= [4 \;\; 15 \;\; -23 \;\; -4 \;\; 8] \\ &= 4\delta[n] + 15\delta[n-1] - 23\delta[n-2] - 4\delta[n-3] + 8\delta[n-4], \\ &\qquad 0 \le n \le 4 \end{aligned}}$$

More Workspace for Problem 1...

Other Way: $X[k]$ and $H[k]$ as before:

$$Y[k] = H[k]X[k]$$

$$= -4 + 2W_5^k + W_5^{2k} - W_5^{3k} - 2W_5^{4k}$$
$$+20W_5^k - 10W_5^{2k} - 5W_5^{3k} + 5W_5^{4k} + 10W_5^{5k}$$
$$-12W_5^{2k} + 6W_5^{3k} + 3W_5^{4k} - 3W_5^{5k} - 6W_5^{6k}$$
$$-4W_5^{3k} + 2W_5^{4k} + W_5^{5k} - W_5^{6k} - 2W_5^{7k}$$

$$Y[k] = -4 + 22W_5^k - 21W_5^{2k} - 4W_5^{3k} + 8W_5^{4k} + 8\underbrace{W_5^{5k}}_{1} - 7\underbrace{W_5^{6k}}_{W_5^k} - 2\underbrace{W_5^{7k}}_{W_5^{2k}}$$

$$= 4 + 15W_5^k - 23W_5^{2k} - 4W_5^{3k} + 8W_5^{4k}$$
$$= \sum_{n=0}^{4} y[n]\, W_5^{nk}$$

$$\boxed{\begin{array}{c} y[n] = [4 \;\; 15 \;\; -23 \;\; -4 \;\; 8] \\ y[n] = 4\delta[n] + 15\delta[n-1] - 23\delta[n-2] - 4\delta[n-3] + 8\delta[n-4], \\ 0 \le n \le 4 \end{array}}$$

2. **25/20 pts**. A digital filter F has transfer function

$$F(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{8}{3}z^{-1}\right)}{\left(1 - \frac{4}{3}e^{j3\pi/4}z^{-1}\right)\left(1 - \frac{4}{3}e^{-j3\pi/4}z^{-1}\right)}, \qquad |z| > \frac{4}{3}.$$

(a) **20/16 pts**. Find the transfer function $H(z)$ for a *new* digital filter H such that:

i. $H(z)$ has the same magnitude response as $F(z)$; i.e., $|H(e^{j\omega})| = |F(e^{j\omega})|$ $\forall$ $\omega \in \mathbb{R}$.

ii. $H(z)$ is causal, stable, and minimum phase.

→ There is a bad zero outside the unit circle at $z = -\frac{8}{3}$.

→ Both poles are bad, outside the unit circle @ $z = \frac{4}{3}e^{\pm j3\pi/4}$

$$F(z) = \underbrace{\left(1-\tfrac{1}{2}z^{-1}\right)}_{\text{good zero}}\underbrace{\left(1+\tfrac{8}{3}z^{-1}\right)}_{\text{Bad Zero}}\underbrace{\frac{1}{\left(1-\frac{4}{3}e^{j3\pi/4}z^{-1}\right)\left(1-\frac{4}{3}e^{-j3\pi/4}z^{-1}\right)}}_{\text{2 bad poles}}$$

$$= \left(1-\tfrac{1}{2}z^{-1}\right)\left(1+\tfrac{8}{3}z^{-1}\right)\left[\frac{z^{-1}+\frac{8}{3}}{z^{-1}+\frac{8}{3}}\right] \leftarrow \text{one}$$

$$\times \frac{1}{\left(1-\frac{4}{3}e^{j3\pi/4}z^{-1}\right)\left(1-\frac{4}{3}e^{-j3\pi/4}z^{-1}\right)}$$

$$\times \underbrace{\frac{\left(z^{-1}-\frac{4}{3}e^{-j3\pi/4}\right)\left(z^{-1}-\frac{4}{3}e^{j3\pi/4}\right)}{\left(z^{-1}-\frac{4}{3}e^{-j3\pi/4}\right)\left(z^{-1}-\frac{4}{3}e^{j3\pi/4}\right)}}_{\text{one}}$$

→

Problem 2, cont…

$$F(z) = \underbrace{\frac{(1-\frac{1}{2}z^{-1})(z^{-1}+\frac{8}{3})}{(z^{-1}-\frac{4}{3}e^{-j3\pi/4})(z^{-1}-\frac{4}{3}e^{j3\pi/4})}}_{H(z)} \underbrace{\frac{(1+8/3z^{-1})(z^{-1}-\frac{4}{3}e^{j3\pi/4}z^{-1}) \times (z^{-1}+\frac{4}{3}e^{-j3\pi/4}z^{-1})}{(z^{-1}+8/3)(1-\frac{4}{3}e^{-j3\pi/4}z^{-1}) \times (1-\frac{4}{3}e^{j3\pi/4}z^{-1})}}_{\text{All Pass}}$$

$$H(z) = \frac{(1-\frac{1}{2}z^{-1})(\frac{8}{3})(1+\frac{3}{8}z^{-1})}{(-\frac{4}{3}e^{-j3\pi/4})(-\frac{4}{3}e^{j3\pi/4})(1-\frac{3}{4}e^{j3\pi/4}z^{-1})(1-\frac{3}{4}e^{-j3\pi/4}z^{-1})}$$

$$= \left(\frac{8}{3}\right)\left(\frac{9}{16}\right)\frac{(1-\frac{1}{2}z^{-1})(1+3/8\,z^{-1})}{(1-\frac{3}{4}e^{j3\pi/4}z^{-1})(1-\frac{3}{4}e^{-j3\pi/4}z^{-1})}$$

$$H(z) = \frac{(\frac{3}{2})(1-\frac{1}{2}z^{-1})(1+3/8\,z^{-1})}{(1-\frac{3}{4}e^{j3\pi/4}z^{-1})(1-\frac{3}{4}e^{-j3\pi/4}z^{-1})}$$

(b) **5/4 pts**. List the poles and zeros of your *new* transfer function $H(z)$ and specify the region of convergence.

Hint: a pole-zero plot is *not* required.

Zeros: $z = \frac{1}{2}, -3/8$

Poles: $z = \frac{3}{4}e^{\pm j3\pi/4}$

ROC: $|z| > 3/4$

3. **25/20 pts**. H is a causal, stable third-order Type II linear phase FIR digital filter. The impulse response $h[n]$ is real and $h[0] = 1$. $H(z)$ has a zero at $z = e^{j2\pi/3}$.

(a) **12/10 pts**. Give a pole-zero plot for $H(z)$.

Hint: It is an FIR filter – don't forget about the poles when you make your plot!

Because $H(z)$ is Type II, there is a real zero at $z=-1$.

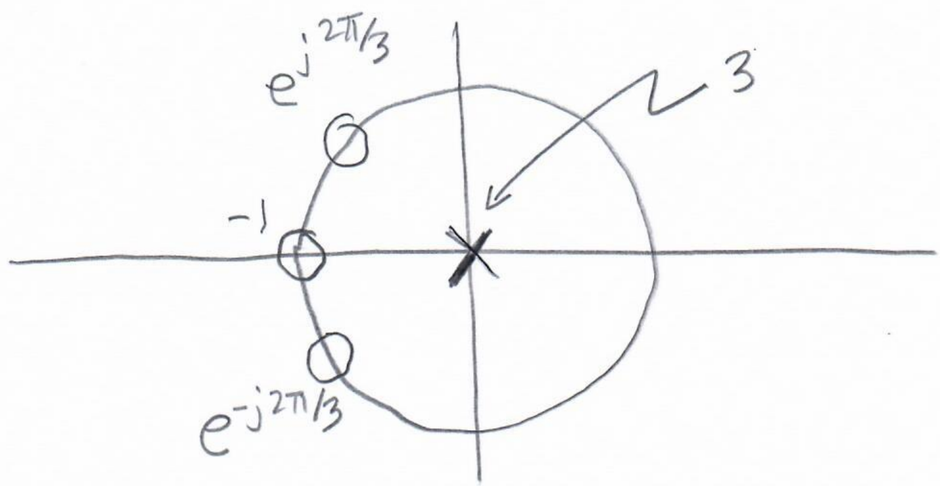
The zero at $z=e^{j2\pi/3}$ is it's own mirror image, but there must be another zero at the conjugate location $z=e^{-j2\pi/3}$.

Since $H(z)$ is FIR, the poles are all at $z=0$ (third-order).

poles: $z=0$ (3rd-order)

zeros: $z=-1,\ e^{\pm j2\pi/3}$

ROC: $|z|>0$ ← not required for full credit answer



Problem 3, cont...

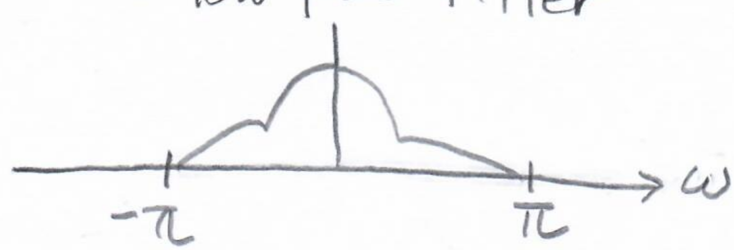
(b) **5/4 pts**. From the pole-zero plot, determine if the filter $H(z)$ is high pass, low pass, band pass, or band stop. *Briefly justify your answer.*

→ A Type II filter cannot be high pass or bandstop because of the zero @ $z = -1$ $(\omega = \pm\pi)$.

→ For this filter, the poles pull $H(z)$ up at $z = 1$ $(\omega = 0)$ and around the unit circle,

→ But the zeros pull $H(z)$ down towards zero on the left half of the unit circle and force $H(z)$ all the way down to zero at $z = -1$ $(\omega = \pm\pi)$.

$\Rightarrow$ $H(z)$ is a low pass filter



(the plot is not required)

Problem 3, cont...

(c) **8/6 pts**. Find the group delay $\tau(\omega)$ for the digital filter $H(z)$.

Short Way: $H(z)$ is a 3rd order linear phase FIR filter. The highest power of $e^{-j\omega}$ in $H(e^{j\omega})$ will be $e^{-j3\omega}$. Factor out $e^{-j3/2\,\omega}$.

$\rightarrow$ Spectral Phase is $\theta(\omega) = -\frac{3}{2}\omega$.

$$\boxed{\tau(\omega) = -\frac{d}{d\omega}\theta(\omega) = \frac{3}{2}}$$

Long Way:

$$H(z) = K(1+z^{-1})(1-e^{j2\pi/3}z^{-1})(1-e^{-j2\pi/3}z^{-1})$$

$$= K(1+z^{-1})(1-[e^{j2\pi/3}+e^{-j2\pi/3}]z^{-1}+z^{-2})$$

$$= K(1+z^{-1})(1-2\cos[2\pi/3]z^{-1}+z^{-2})$$

$$= K(1+z^{-1})(1+z^{-1}+z^{-2})$$

$$= K(1+z^{-1}+z^{-2} + z^{-1}+z^{-2}+z^{-3})$$

$$= K(1+2z^{-1}+2z^{-2}+z^{-3}),\ |z|>0$$

$$h[n] = K\delta[n] + 2K\delta[n-1] + 2K\delta[n-2] + K\delta[n-3]$$

$$h[0] = 1 \mapsto K = 1$$

$$H(z) = 1+2z^{-1}+2z^{-2}+z^{-3},\ |z|>0$$

$$H(e^{j\omega}) = 1+2e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}$$

$$= [e^{j\frac{3}{2}\omega} + 2e^{-j\frac{1}{2}\omega} + 2e^{j\frac{1}{2}\omega} + e^{-j3/2\,\omega}] \times e^{-j3/2\,\omega}$$

$$= [2\cos(\tfrac{3}{2}\omega) + 4\cos(\tfrac{1}{2}\omega)] \times e^{-j3/2\,\omega}$$

$$A(\omega) = 2\cos(\tfrac{3}{2}\omega) + 4\cos(\tfrac{1}{2}\omega)$$

$$\theta(\omega) = -\frac{3}{2}\omega$$

$$\downarrow$$

$$\boxed{\tau(\omega) = -\frac{d}{d\omega}\theta(\omega) = 3/2}$$

Numbers in boxes are calculator registers

4. **25/20 pts**. Design an **analog** Type I Chebyshev low pass filter to meet the following analog design specification:

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{9}{10}$$

$$\sqrt{1+\varepsilon^2} = 10/9$$

$$1+\varepsilon^2 = 100/81$$

$$\varepsilon^2 = \frac{100-81}{81} = 19/81$$

$$\varepsilon = \sqrt{19}/9$$

$$\boxed{05}\ \varepsilon = 0.484322$$

$$\frac{1}{A} = \frac{1}{\pi} \qquad A = \pi$$

Passband Edge Freq.	$\Omega_p = \pi$ rad/sec
Stopband Edge Freq.	$\Omega_s = 2\pi$ rad/sec
Min. Stopband Attenuation	$1/A = 1/\pi$
Max. Passband Attenuation	$1/\sqrt{1+\varepsilon^2} = 0.90$

Give the analog filter transfer function $H_a(s)$.

Hint: you can leave $H_a(s)$ in factored form – i.e., you do **not** have to "multiply out" the numerator and denominator.

$$N = \left\lceil \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\cosh^{-1}(\sqrt{\pi^2-1}/\varepsilon)}{\cosh^{-1}(2)} \right\rceil = \left\lceil \frac{\cosh^{-1}(\boxed{06}\ 6.14919)}{\boxed{07}\ 1.31696} \right\rceil$$

$$= \left\lceil \frac{2.50279}{1.31696} \right\rceil = \lceil 1.90043 \rceil = 2 //$$

$$\gamma = \left(\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)^{\frac{1}{N}} = \left(\frac{1+10/9}{\varepsilon}\right)^{1/2} = \boxed{08}\ 2.08780$$

$$\zeta = \frac{\gamma^2+1}{2\gamma} = \boxed{09}\ 1.28339 \qquad \xi = \frac{\gamma^2-1}{2\gamma} = \boxed{10}\ 0.804412$$

$$\sigma_1 = -\Omega_p \xi \sin\left[\frac{(2-1)\pi}{2\cdot 2}\right] = -\pi\xi\sin\frac{\pi}{4} = \boxed{11}\ -1.78695$$

$$\sigma_2 = -\Omega_p \xi \sin\left[\frac{(4-1)\pi}{2\cdot 2}\right] = -\pi\xi\sin\frac{3\pi}{4} = -1.78695 = \sigma_1$$

$$\Omega_1 = \Omega_p \zeta \cos\left[\frac{(2-1)\pi}{2\cdot 2}\right] = \pi\zeta\cos\frac{\pi}{4} = 2.85097\ \boxed{12}$$

$$\Omega_2 = \Omega_p \zeta \cos\left[\frac{(4-1)\pi}{2\cdot 2}\right] = \pi\zeta\cos\frac{3\pi}{4} = -2.85097 = -\Omega_1\ \boxed{13}$$

$$p_1 = \sigma_1 + j\Omega_1 = -1.78695 + j2.85097$$

$$p_2 = \sigma_2 + j\Omega_2 = -1.78695 - j2.85097$$

$$N \text{ is even} \rightarrow C_0 = \frac{1}{\sqrt{1+\varepsilon^2}} = \frac{9}{10} \qquad \longmapsto$$

More Workspace for Problem 4...

$$H_a(s) = C_0 \prod_{\ell=1}^{2} \frac{-p_\ell}{s-p_\ell} = C_0 \frac{(-p_1)(-p_2)}{(s-p_1)(s-p_2)}$$

$$= C_0 \frac{p_1 p_2}{(s-p_1)(s-p_2)} = C_0 \frac{p_1 p_1^*}{(s-p_1)(s-p_1^*)}$$

$$= \frac{\frac{9}{10}|p_1|^2}{(s-p_1)(s-p_1^*)}$$

$$\boxed{H_a(s) = \frac{10.1891}{(s+1.78695-j2.85097)(s+1.78695+j2.85097)}}$$

Full Credit Answer

$$= \frac{C_0 p_1 p_1^*}{s^2-(p_1+p_1^*)s+p_1 p_1^*} = \frac{C_0|p_1|^2}{s^2-2\sigma_1 s+|p_1|^2}$$

$$= \frac{C_0 \sigma_1^2 \Omega_1^2}{s^2-2\sigma_1 s+\sigma_1^2\Omega_1^2}$$

$$\boxed{H_a(s) = \frac{10.1891}{s^2+3.57391s+11.3212}}$$

$Re[s] > \sigma_1 = -1.78695$

Numbers in boxes are calculator registers

5. **25/20 pts**. Design a causal, stable "simple" second-order IIR digital bandstop filter with a notch frequency of $\omega_0 = \frac{\pi}{3}$ rad/sample and a quality of $Q = 4$.

Give the transfer function $H(z)$ and be sure to specify the ROC.

Hint 1: make sure that your calculator is set for **radians** and **not** degrees!

Hint 2: the bandpass and bandstop filters are discussed on notes pages 7.58 and 7.59. They both have the same denominator for $H(z)$ – so they both have the same poles. Only the numerators (and hence the zeros) are different. In other words, you find the poles and ROC for the bandstop filter exactly the same way you do it for the bandpass filter.

$$\omega_0 = \frac{\pi}{3} = \arccos\beta \longmapsto \beta = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$Q = 4 = \frac{\omega_0}{B_W} \longmapsto B_W = \frac{\omega_0}{4} = \frac{\pi}{3}\cdot\frac{1}{4} = \pi/12$$

$$B_W = \arccos\left[\frac{2\alpha}{1+\alpha^2}\right] \longmapsto \frac{2\alpha}{1+\alpha^2} = \cos\frac{\pi}{12}$$

$$2\alpha = \cos\left(\tfrac{\pi}{12}\right) + \cos(\pi/12)\,\alpha^2$$

$$\left[\cos\tfrac{\pi}{12}\right]\alpha^2 - 2\alpha + \left[\cos\tfrac{\pi}{12}\right] = 0$$

$a = \cos\frac{\pi}{12} = 0.965926$ $\boxed{51}$

$b = -2$

$c = \cos\frac{\pi}{12}$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4\cos^2 \pi/12}}{2\cos\pi/12}$$

$$= \frac{2 \pm \sqrt{0.267949}\ \boxed{52}}{1.93185\ \boxed{53}} = \frac{2 \pm 0.517639\ \boxed{54}}{1.93185}$$

$\boxed{55}$

$$\alpha = 1.30323 \quad \underline{\text{or}} \quad \alpha = 0.767327$$

For stability, we must choose $|\alpha| < 1$

$$\Rightarrow \alpha = 0.767327 \ //$$

More Workspace for Problem 5...

Two poles at $z = re^{\pm j\varphi}$

where $r = \sqrt{\alpha} = 0.875972$ $\boxed{56}$

$$H(z) = \frac{1+\alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha) z^{-1} + \alpha z^{-2}}$$

$$= \frac{1+\alpha}{2} \frac{1 - z^{-1} + z^{-2}}{1 - \frac{1}{2}(1.767327) z^{-1} + 0.767327 z^{-2}}$$

$$\boxed{H(z) = \frac{0.883663(1 - z^{-1} + z^{-2})}{1 - 0.883663 z^{-1} + 0.767327 z^{-1}} \qquad |z| > 0.875972}$$