ECE 4213/5213
Homework 5

Fall 2023 Dr. Havlicek

1. Consider a continous-time LTI Gaussian filter H with impulse response

$$h(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}},$$

where $\sigma, \mu \in \mathbb{R}$ are constants.

   (a) Find the filter frequency response $H(\Omega) = \mathcal{F}\{h(t)\}$. **Hint:** Carefully set up the Fourier transform integral. Pull out the constants and combine all the exponentials that remain under the integral. This will leave you with an integrand that is an exponential where the exponent is a quadratic polynomial with complex coefficients. You should be able to find this definite integral in your math handbook (or any other good table of definite integrals) and just write down the answer.

   (b) Find the spectral magnitude $A(\Omega)$ and spectral phase $\theta(\Omega)$. **Hint:** the spectral magnitude and spectral phase for a DTFT $X(e^{j\omega})$ are defined on page 3.99 of the notes. For the continous Fourier transform they are defined analogously; i.e., $H(\Omega) = A(\Omega)e^{j\theta(\Omega)}$.

   (c) Find the group delay of the filter. **Hint:** group delay is defined on page 3.120 of the notes.

   (d) Can you find conditions on the constants σ, μ to guarantee that the impulse response $h(t)$ and frequency response $H(\Omega)$ will both be Gaussian functions?

2. Prove the Fourier transform time shift property, i.e.

$$\text{if } x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega), \text{ then } x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega)e^{-j\Omega t_0}.$$

3. Prove the Fourier transform time differentiation property, i.e.,

$$\text{if } x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega), \text{ then } \frac{d}{dt}x(t) \overset{\mathcal{F}}{\longleftrightarrow} j\Omega X(\Omega).$$

4. Text problem 2.6(b).

5. Text problem 2.8.

6. Text problem 2.9(a).

7. Text problem 2.11. **Hint:** apply Euler's formula to write the sine as a sum of two complex exponentials and then use the techniques illustrated on pages 3.113 - 3.114 of the notes.

8. Text problem 2.13, parts (a), (b), and (d) only.
9. Text problem 2.33(a) and (b). **Note:** in part (b), the "delay" n_d is equal to the group delay $\tau(\omega)$.
10. Text problem 2.34(b).
11. Find the discrete-time signals $x_1[n]$ and $x_2[n]$ with DTFT's given by:

    (a)
    $$X_1(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{j\omega}}$$
    Hint: use the time reversal property.

    (b)
    $$X_2(e^{j\omega}) = \frac{-3e^{-j\omega}}{1 - 3e^{-j\omega}}$$
    Hint: your first instinct may be to try applying the formula for the DTFT of $a^n u[n]$ directly (followed by a time shift to account for the numerator). But that won't work – because the formula for $a^n u[n]$ requires $|a| < 1$. Instead, try multiplying $X_2(e^{j\omega})$ by $1 = \alpha/\alpha$ where $\alpha = -\frac{1}{3}e^{j\omega}$ and then apply the time reversal property.

DUE: 10/11/2023, 11:59 PM

(Scan or photograph your paper and upload to Canvas)